3 1761 05476662 1

$\sqrt{2} = 1.4142136$

$\sqrt{3} = 1.7320508$

$\sqrt{4} = 2.$

$\sqrt{5} = 2.2360679$? = 2.2360680 nearly

$\sqrt{6} = 2.4494897$

$\pi = 3.1415926535 = \frac{355}{113}$ nearly

Edm. John Senkler

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MatA
W951s

SELF-EXAMINATIONS

IN

ALGEBRA.

BY

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ADVERTISEMENT.

THIS work will be deemed highly useful as a Praxis to Students in general, but particularly so to such of them as cannot procure the helps of a Tutor. It has no claim to originality, being merely a work written by that indefatigable and most useful author, Meier Hirsch, of Berlin, anglicized and adapted to the taste of the British Mathematicians. The author, therefore, claims no merit beyond that of having catered well for them.

London, May 10, 1825.

SECTION I.

CONTAINING

EXAMPLES FOR THE GENERAL RULES OF CALCULATION.

I. DECIMAL FRACTIONS.

WHAT is meant by a Decimal Fraction? What change does it undergo, when the Decimal Point is moved one place backwards or forwards? How is a Vulgar Fraction converted into a Decimal Fraction? What is a Period in Decimal Fractions? When a Vulgar Fraction is transformed into a Decimal, how many Digits can such a Period then contain at least? It being allowed in the Products and Quotients to omit the last Decimal Digits as unimportant compared with the preceding ones, may the Abridged Multiplication, and Abridged Division be used? Wherein consists their advantage, and whence its source?

1. ADDITION.

1.	0·857	2.	1·007	3.	0·00076
	0·678		2·346		13·795
	1·535		3·353		13·79576

4. 12·0134
196·785
7·00006
215·79846

5. 0·90058
7·634
3·007956
11·542536

6. 7·345
8·26
37·534
19·0005
10·94
103·729
186·8085

7. 7·6
138·05934
15·4
10·76
0·3592176
1365·7
37·6483
0·005
1575·5318576

8. 19·3576
17·2340
7652·007
0·5
39·069534
7·83
5·69784
2·350006
7744·04598

9. 113·67849
76·859
9·7
5
152·6043
7·85976
9·437
8·65
7·94
391·72855

2. SUBTRACTION.

1. 0·947
0·195
0·752

2. 9·567
3·078
6·489

3. 213·5734
87·6572
125·9162

4.	54·763	5.	73·5673	6.	385·76943
	0·921		12 889		72·57
	53·842		60 6783		313·19943
7.	27·003	8.	129·57	9.	0·975
	7·6854		6·894356		0·483764
	19·3176		122·675644		0·491236
10.	23·005	11.	96·5	12.	0·5
	4·76943		0·000783		0·0003
	18·23557		96·499217		0·4997

3. MULTIPLICATION.

1. $3·57 \times 6 = 21·42$
2. $5·798 \times 18 = 104·364$
3. $0·5 \times 36000 = 18000$
4. $0·00563 \times 17 = 0·09571$
5. $0·0000054 \times 3785 = 0·020439$
6. $3·7 \times 2·6 = 9·62$
7. $5·78 \times 3·4 = 19·652$
8. $3·9765 \times 4·378 = 17·409117$
9. $32·76859 \times 13·0076 = 426·24071128$
10. $138·5 \times 7·695708 = 1065·855558$
11. $0·43 \times 0·65 = 0·2795$
12. $0·576 \times 0·3854 = 0·2219904$
13. $0·005 \times 0·017 = 0·000085$
14. $0·007853 \times 0·00476 = 0·00003738028$
15. $113·5 \times 0·072 = 8·172$
16. $0·372106 \times 0·0054 = 0·0020093724$
17. $0·137 \times 0·00056 = 0·00007672$
18. $0·376 \times 0·0076894 = 0·0028912144$

Abridged Multiplication.

1. 7·65340958 × 2·56307

1530681916
382670479
45920457
2296022
53573
19·61622447

2. 0·7653478 × 0·3576

22960434
3826739
535742
45920
0·27368835

3. 8·56794323 × 0·5284765

4283971615
171358864
68543545
3427177
599755
51407
4283
4·527956646

4. 37·346859416 × 0·007003458

261428015912
112040578
14938743
1867342
298774
0·261557161349

5. 0·076934210834 × 0·000003057026

230802632502
3849710541
538539475
1548684
461605
0·000000235289882807

6. 15·7356783 × 2·564725

314713566
78678391
9441406
629426
141620
3147
786
40·3608342

4. DIVISION.

1. $5{\cdot}64 : 2 = 2{\cdot}82$
2. $7{\cdot}5832 : 8 = 0{\cdot}9479$
3. $0{\cdot}357642 : 6 = 0{\cdot}059607$
4. $0{\cdot}32769414 : 18 = 0{\cdot}01820523$
5. $0{\cdot}01765125 : 375 = 0{\cdot}00004707$
6. $75 : 16 = 4{\cdot}6875$
7. $731 : 8 = 91{\cdot}375$
8. $3{\cdot}54 : 7 = 0{\cdot}50571428$ .... (period 571428)
9. $8{\cdot}2356 : 17 = 0{\cdot}484447$...(period 4705882352941176)
10. $273{\cdot}694 : 543 = 0{\cdot}50404051$...
11. $6938{\cdot}57 : 276 = 25{\cdot}13974637$....
12. $0{\cdot}000215 : 316 = 0{\cdot}0000006803797$...
13. $400 : 0{\cdot}25 = 1600$
14. $378 : 0{\cdot}01 = 37800$
15. $5640 : 0{\cdot}0015 = 3760000$
16. $183260 : 0{\cdot}476 = 385000$
17. $1 : 0{\cdot}24 = 4{\cdot}16666$...
18. $2{\cdot}53944 : 7{\cdot}2 = 0{\cdot}3527$
19. $0{\cdot}02382245 : 0{\cdot}37 = 0{\cdot}064385$
20. $1114{\cdot}869145005 : 0{\cdot}385 = 2895{\cdot}764013$
21. $56{\cdot}4 : 0{\cdot}00015 = 376000$
22. $10287{\cdot}36 : 0{\cdot}0036 = 2857600$
23. $0{\cdot}0001 : 0{\cdot}02 = 0{\cdot}005$
24. $145{\cdot}817 : 0{\cdot}0563 = 2590$
25. $374 : 2{\cdot}4 = 155{\cdot}833333$...
26. $15{\cdot}713 : 18{\cdot}13 = 0{\cdot}86668505$...
27. $137{\cdot}51634 : 27{\cdot}65 = 4{\cdot}97346618$...
28. $0{\cdot}5 : 76{\cdot}191342 = 0{\cdot}00650081$...
29. $0{\cdot}046 : 0{\cdot}00762089 = 6{\cdot}0360404$...
30. $1 : 3{\cdot}2561047 = 0{\cdot}30711543$...
31. $38076 : 137 = 277.92700729$...

Abridged Division.

1.

3·716048	
7·632035	2·053804...
7·432096	
199939	
185802	
14137	
11148	
2989	
2972	
17	
14	
3	

2.

15·314865	
2·0000000	0·13059209...
1·5314865	
4685135	
4594459	
90676	
76574	
14102	
13782	
320	
306	
14	
13	
1	

3.

0·3547808	
10·926954	30·79917...
10·643424	
283530	
248346	
35184	
31930	
3254	
3192	
62	
35	
27	
24	
3	

4.

0·00358432 97	
3·00000000	836·97659...
2·86746376	
13253624	
10752989	
2500635	
2150597	
350038	
322588	
27450	
25090	
2360	
2150	
210	
179	
31	
31	

5. TRANSFORMATION OF VULGAR FRACTIONS TO DECIMAL FRACTIONS.

1. $\frac{1}{2} = 0{\cdot}5$
2. $\frac{3}{4} = 0{\cdot}75$
3. $\frac{13}{16} = 0{\cdot}8125$
4. $\frac{17}{20} = 0{\cdot}85$
5. $\frac{3}{40} = 0{\cdot}075$
6. $\frac{17}{125} = 0{\cdot}136$
7. $\frac{7}{800} = 0{\cdot}00875$
8. $\frac{372}{1250} = 0{\cdot}2976$
9. $\frac{11}{16000} = 0{\cdot}0006875$
10. $\frac{15}{1280} = 0{\cdot}01171875$
11. $\frac{347}{25600} = 0{\cdot}0135546875$
12. $\frac{1}{10000} = 0{\cdot}0001$
13. $\frac{3476}{15625} = 0{\cdot}222464$
14. $\frac{1}{2048000} = 0{\cdot}00000048828125$
15. $\frac{2}{3} = 0{\cdot}666666....$
16. $\frac{3}{7} = 0{\cdot}4285714....$ (period 428571)
17. $\frac{15}{17} = 0.88235....$ (period 8823529411764705)
18. $\frac{3}{35} = 0{\cdot}0857142857....$
19. $\frac{17}{19} = 0{\cdot}894736842105263157....$ (period)
20. $\frac{19}{1385} = 0{\cdot}0139194139....$
21. $\frac{706}{3907} = 0{\cdot}1195192144....$
22. $\frac{1}{679} = 0{\cdot}0014727540....$
23. $\frac{1}{8076934} = 0{\cdot}0000001238....$
24. $\frac{13}{339437} = 0{\cdot}0000228295....$
25. $\frac{130}{2769135} = 0{\cdot}0000469460....$
26. $\frac{17}{30000} = 0{\cdot}00056666....$
27. $\frac{47}{71000000} = 0{\cdot}0000006619....$

1. What part of a year is a day, calculating the year at 365 days and 6 hours? *Ans.* 0·0027378507....

2. But what part is it, when the year, as is right, is computed to average 365·2422453 days?

Ans. 0·0027379034....

3. We divide the circumference of a circle into 360 degrees, the degree into 60 minutes, the minute into 60 seconds; the French, however, divide it into 400 degrees, and express the smaller portions by decimals of the degree. How much then do our degree, minute, and second amount to in French degrees?

Ans. $1^\circ = 1^\circ.11111....$ French, $1' = .018518....$ deg. French, and $1'' = .000308....$ deg. French.

4. What do $57^\circ.9467$ of the French division amount to in our degrees, minutes, and seconds?

Ans. $52^\circ.9'.7''.308.$

5. But what do $43^\circ.6'.20''$ of our division come to by the French system? *Ans.* $47^\circ.895....$

6. What is the fourth proportional to the numbers 0·45, 0·8, and 0·367? *Ans.* 0·652444....

7. A cubic inch of the purest gold, weighs about 19·64 times as much as a cubic inch of distilled water; but a cubic inch of Japan copper only 9 times as much. How large must a piece of Japan copper be, to weigh as much as $\frac{3}{4}$ cubic inch of the gold?

Ans. 1·6366.... cubic inches.

II. ALGEBRA.

THE great progress of this science during the last century; the consequent increase and intricacy of its rules and expressions, and the limited power of our minds, soon made it necessary to introduce a language of symbols more comprehensive, expressive, and determinate, than mere words. How does Algebra or the Literal Calculus effect this object? What signs are used for Addition, Subtraction, Multiplication, and Division? What does an Aggregate of magnitudes infer? What signify Brackets or Crotchets? In what cases are they used? Would the omission of them give rise to a mistake? What are Coefficients?

What do we term opposite or contrary magnitudes? What mean *Positive* and *Negative*, or *Affirmative* and *Primative*? That which arises from the magnitudes being opposite or contrary, is only a *quality*—a reference of the one to the other. What is a magnitude without such reference? The Addition of opposite magnitudes is the Subtraction of absolute ones: Why?

ADDITION.

(a) *Addition of Simple Magnitudes.*

1.	2.	3.	4.
a	$7a$	$8f$	a
a	$5a$	f	b
$2a$	$12a$	$9f$	$a+b$

5. $7a$ / $10x$ / $7a+10x$

6. a / $-a$ / 0

7. $17a$ / $-6a$ / $11a$

8. $5a$ / $-9a$ / $-4a$

9. $-6a$ / $10a$ / $4a$

10. $-3a$ / $2a$ / $-a$

11. a / $-b$ / $a-b$

12. $8a$ / $-5b$ / $8a-5b$

13. $-7a$ / b / $-7a+b$ Or $b-7a$

14. $-a$ / $-a$ / $-2a$

15. $-3a$ / $-8a$ / $-11a$

16. $-8a$ / $-3b$ / $-8a-3b$ Or $(8a+3b)$

17. $3a$ / $-5a$ / $8a$ / $6a$

18. $-12b$ / $-4b$ / $13b$ / $-3b$

19. $-6f$ / $9f$ / $13f$ / $-8f$ / $8f$

20. $-7c$ / $-5c$ / $-3c$ / $-15c$

21. $5d$ / $7d$ / d / $-8d$ / $5d$

22. $8e$ / $-4e$ / $7e$ / $-11e$ / 0

23. $3g$ / $2h$ / $-5g$ / $7h$ / $-10g$ / $-12g+9h$ Or $9h-12g$

24. $3a$ / $-5a$ / $7a$ / $2b$ / $-c$ / $4b$ / $5a+6b-c$

(b) *Addition of Compound Magnitudes.*

1. $7a-5c+3b$ / $2a-3c-7b$ / $9a-8c-4b$

2. $5a+4b-3c-7d+8$ / $3a-12b+7c-10d-4$ / $8a-8b+4c-17d+4$

3. $12h - 3c - 7f + 3g$
$-3h + 8c - 2f - 9g + 5x$
$9h + 5c - 9f - 6g + 5x$

4. $16a - 5b + 10c - 9d$
$3a + 18b - 5c - 7d + 3e$
$-7a - 2b - 3d + 5e - 9h$
$11a - 3b + 2c + 8d + 7h$
$23a + 8b + 7c - 11d + 8e - 2h$

5. $7x - 6y + 5z + 3 - g$
$-x - 3y - 8 - g$
$-x + y - 3z - 1 + 7g$
$-2x + 3y + 3z - 1 - 3$
$x + 8y - 5z + 9 + g$
$4x + 3y + 2 + 5g$

6. $8a + b$
$2a - b + c$
$-3a + 5b + 2d$
$-6b - 3c + 3d$
$-5a + 7c - 2d$
$2a - b + 5c + 3d$

7. $-a + 3b - c - 115d + 6e - 5f$
$3a - 2b - 3c - d + 27e$
$5b - 8c + 3e - 7f$
$-7a - 6b + 17c + 9d - 5e + 11f$
$-3a - 5c - 2d + 6e - 9f + g$
$-8a - 109d + 37e - 10f + g$

8. $-7f + 3a$
$4f - 2a$
$3f - 3a$
$+2a$
0

SUBTRACTION.

(a) *Subtraction of Simple Magnitudes.*

1.	2.	3.	4.
a	$7a$	$10f$	$5d$
a	$3a$	$10f$	$11d$
0	$4a$	0	$-6d$

5.	6.	7.	8.
$7a$	a	$8a$	$6a$
$5b$	$-a$	$-a$	$-5a$
$7a-5b$	$2a$	$9a$	$11a$

9.	10.	11.	12.
a	a	$3a$	$-9a$
$-4a$	$-b$	$-2b$	$3a$
$5a$	$a+b$	$3a+2b$	$-12a$

13.	14.	15.	16.
$-7a$	$-19a$	$-6a$	$-3a$
$-7a$	$-20a$	$-5a$	$-5b$
0	a	$-a$	$-3a+5b$
			Or $5b-3a$

17.	18.	19.	20.
-13	-8	12	-13
3	-17	-7	-8
-16	9	19	-5

(b) *Subtraction of Compound Magnitudes.*

1.	2.
$3a-2b+6$	$13a-2b+9c-3d$
$2a-7b-3$	$8a-6b+9c-10d+12$
$a+5b+9$	$5a+4b+7d-12$

3.	4.
$-7f+3m-8x$	$2a-c-h-l$
$-6f-5m-2x+3d+8$	$9a-3e+4h-l-c$
$-f+8m-6x-3d-8$	$-7a+2e-5h+c$

5.	6.
$-a-5b+7c-d$	$3h-2k$
$4b-3c+2d+3k$	$9l-7-8k$
$-a-9b+10c-3d-3k$	$3h-9l+6k+7$

7. $-3a+b-8c+7e-\ 5f+3h-7x-13y$
$k+2a\quad -9c+8e+\ 7f\quad -7x-\ y-3l-k$

$-5a+b+\ c-\ e-12f+3h\quad -12y+3l$

8. $5b-3a+203c+5$
$-2b-8a+\ 67c+7$

$7b+5a+136c-2$

9. $-14b+3c-27d+\ 3-\ 5g$
$7a-\ 5c-8d+\ 3b-12+\ 7g$

$-7a-17b+8c-19d+15-12g$

10. $6a+\ 5-3b-\ 5f-\ g-\ h$
$-2a-9b+8g-\ 9h+7f-\ 8$

$8a+6b+13-12f-9g+8h$

11. $3c-2l+5c$
$8l+7c-4l$

$c-6l$

12. $3a-17b-10b+13a-2a$
$6b-\ 8a-\ b-\ 2a+3d+9a-5h$

$15a-32b-\ 3d+5h$

13. $5c+3$
$2c-9-\ 7c$

$10c+12$

14. $8a-5b-3c-7d+5c-8f+3g+17k$
$-2k+3c-5b+2d-4e-7f+9g-\ 5k-l$

$8a-6c\quad -9d+9e-\ f-6g+24k+l$

15. $32a+3b-(5a+17b)=27a-14b$

16. $13a-(5c+3f-7a-5x+3a)=17a-5c-3f+5x$

17. $-8a+5b-3c-(7a-3b-2c)=-15a+8b-c$

18. $3a-5c+3d-(7a-5d+8c-2e)=8d+2e-4a-13c$

19. $37a-5f-(3a-2b-5c)-(6a-4b+3h)$
$=28a+6b-5f+5c-3h$

20. $a+b-(2a-3b)-(5a+7b)-(-13a+2b)=7a-5b$

21. $\frac{1}{2}a-\frac{5}{6}x-(\frac{3}{4}a-\frac{1}{2}x)-(3b+\frac{11}{4}x-\frac{2}{3}a)$
$=\frac{5}{12}a-\frac{37}{12}x-3b=\dfrac{5a}{12}-\dfrac{37x}{12}-3b$

MULTIPLICATION.

(a) *Multiplication of Simple Magnitudes.*

1. $a \times b = b \times a = ab = ba = a \cdot b = b \cdot a$
2. $a \times b \times c = abc = acb = bac = bca = cab = cba$
3. $-a \times b = -ab$
4. $a \times -b = -ab$
5. $-a \times -b = ab$
6. $6a \times 7b = 42ab$
7. $17a \times \frac{3}{4}b = \frac{51}{4}ab = \frac{51ab}{4}$
8. $\frac{3a}{2} \times \frac{5f}{4} = \frac{15af}{8} = \frac{15}{8}af$
9. $-3a \times 14c = -42ac$
10. $7a \times -10b = -70ab$
11. $\frac{3}{8}a \times -\frac{2}{5}b = -\frac{1}{4}ab = -\frac{ab}{4}$
12. $a \times -7b = -7ab$
13. $-6a \times -11x = 66ax$
14. $-\frac{5a}{4} \times -\frac{3b}{7} = \frac{15ab}{28}$
15. $ab \times cde = abcde$
16. $-5abc \times 7ade = -35a^2bcde$
17. $-5bd \times 9b^2dxy = -45b^3d^2xy$
18. $-\frac{1}{2}abd \times \frac{1}{4}cdef = -\frac{1}{8}abcd^2ef$
19. $17ace \times 5 = 85ace$

20. $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

21. $-\frac{3fh}{5cde} \times 4g = -\frac{3afgh}{5cde}$

22. $\frac{1}{fgh} \times 4cd = \frac{4cd}{fgh}$

23. $\frac{5ac}{bde} \times -\frac{7bfg}{3ad} = -\frac{35cfg}{3d^2e}$

24. $-\frac{h}{2fg} \times -\frac{g}{3h} = \frac{1}{6f}$

25. $3ab \times 2cd \times dfg = 6abcd^2fg$

26. $-\frac{a}{b} \times b \times -5df = 5adf$

27. $-4ab \times -\frac{3cde}{2a^2b} \times -\frac{1}{5cf} = -\frac{6de}{5af}$

28. $-a \times -a \times -a \times -a = a^4$

29. $\frac{2ac}{bg} \times -\frac{3bd}{4cfh} \times hf = -\frac{3ad}{2g}$

30. $\frac{1}{a} \times \frac{3ac}{x} \times \frac{c}{x} = \frac{3c^2}{x^2}$

31. $\frac{2fg}{bdh} \times \frac{3xy}{7fz} \times -\frac{bc}{5d} = -\frac{6cgxy}{35d^2hz}$

(b) *Multiplication of Compound Magnitudes.*

1. $(6a + 3b - 5f) \times 5g = 30ag + 15bg - 25fg$
2. $(-2b + 3c - g) \times -8h = 16bh - 24ch + 8gh$
3. $(7ad - 16bc - 16acf) \times 10ab = 70a^2bd - 150ab^2c - 160a^2bcf$

4. $\left(\frac{5a}{b} - \frac{13c}{2d} - \frac{6h}{5bg} + 7d\right) \times \frac{3a}{5d} = \frac{3a^2}{bd} - \frac{39ac}{10d^2} - \frac{18ah}{25bdg} + \frac{21a}{5}$

5. $(a + b)(c + d) = ac + bc + ad + bd$

6. $(a + b - c)(d - e) = ad + bd - cd - ae - be + ce$

7. $(2a - 3b - 8c - d + 9e) \times (7f + 2g - h) = 14af - 21bf - 56cf - 7df + 63ef + 4ag - 6bg - 16cg - 2dg + 18eg - 2ah + 3bh + 8ch + dh - 9eh$

8. $(7l - 2m - 9) \times (3l - 11m) = 21l^2 - lm - 27l + 22m^2 + 99m$

9. $(2a + 5b + 3c - 5e) \times (3a + 10b + 15f) = 6a^2 + 35ab + 9ac - 15ae + 50b^2 + 30bc - 50be + 30af + 75bf + 45cf - 75ef$

10. $(a + b) \times (a - b) = a^2 - b^2$

11. $(a + b) \times (a + b) = a^2 + 2ab + b^2$

12. $(a - b) \times (a - b) = a^2 - 2ab + b^2$

13. $(3a + 5b - \frac{7}{2}c) \times (a - 2b + 9c) = 3a^2 - ab + \frac{47}{2}ac - 10b^2 + 52bc - \frac{63}{2}c^2$

14. $(3c - 5d + \frac{5}{4}g - \frac{5}{3}h) \times (\frac{2}{3}c - d + 7g + \frac{1}{2}h) = 2c^2 - \frac{19}{3}cd + \frac{43}{2}cg + \frac{7}{18}ch + 5d^2 - \frac{145}{4}dg - \frac{5}{6}dh + \frac{21}{4}g^2 - \frac{271}{24}gh - \frac{5}{6}h^2$

15. $(5ab + 3ac - 4bc) \times (7ab - 18ac + 2bc + d) = 35a^2b^2 - 69a^2bc - 18ab^2c + 5abd - 54a^2c^2 + 78abc^2 + 3acd - 8b^2c^2 - 4bcd$

16. $(13bcd + 20bce - 10bde) \times (4bc + 3bd - 12e) = 52b^2c^2d + 80b^2c^2e + 20b^2cde + 39b^2cd^2 - 30b^2d^2e - 156bcde - 240bce^2 + 120bde^2$

17. $(5a^2 - 3ab + 7b^2) \times (3a - b) = 15a^3 - 14a^2b + 24ab^2 - 7b^3$

18. $(a + b + c) \times (a + b - c)$ $\sqrt{a^2 + 2ab} \sqrt{b^2 - c}$?

$= a^2 + 2ab + b^2 - c^2$

19. $(3a^3 + 35a^2b - 17ab^2 - 13b^3) \times (3a^2 + 26ab - 57b^2) = 9a^5 + 183a^4b + 688a^3b^2 - 2476a^2b^3 + 631ab^4 + 741b^5$

20. $(3a - b + 2c - 3d + 5e) \times (17a - 2b + 12c) = 51a^2 - 23ab + 70ac - 51ad + 85ae + 2b^2 - 16bc + 6bd - 10be + 24c^2 - 36cd + 60ce$

21. $(a + b + c + d) \times (a - b - c - d) = a^2 - b^2 - 2bc - 2bd - c^2 - 2cd - d^2$

22. $(-2a + 3b - c^2) \times (-3f - 7a + c^2) = 6af - 9bf + 3c^2f + 14a^2 - 21ab + 5ac^2 + 3bc^2 - c^4$

23. $(\frac{3}{2}m - 5n - \frac{1}{3}p^2) \times (\frac{1}{4}m - 2n + 6p^2) = \frac{3}{8}m^2 - \frac{17}{4}mn + \frac{107}{12}mp^2 + 10n^2 - \frac{88}{3}np^2 - 2p^4$

24. $\left(\frac{5f^2}{g} - \frac{7fg}{4h} + 3f\right) \times \left(\frac{7g}{f} + \frac{2f}{g}\right) = 35f - \frac{49g^2}{4h} + 21g + \frac{10f^3}{g^2} - \frac{7f^2}{2h} + \frac{6f^2}{g}$

25. $\left(\frac{a^2}{x^2} - \frac{ab}{2xy} + \frac{b^2}{y^2}\right) \times \left(\frac{3a^2}{x^2} - \frac{2ab}{5xy} + \frac{b^2}{y^2}\right) = \frac{3a^4}{x^4} - \frac{19a^3b}{10x^3y} + \frac{21a^2b^2}{5x^2y^2} - \frac{9ab^3}{10xy^3} + \frac{b^4}{y^4}$

DIVISION.

(a) *Division of Simple Magnitudes.*

1. $a : b = \frac{a}{b}$
2. $-a : b = -\frac{a}{b}$
3. $a : -b = -\frac{a}{b}$
4. $-a : -b = \frac{a}{b}$
5. $5a : 3b = \frac{5a}{3b}$
6. $-16a : 8b = -\frac{2a}{b}$
7. $12a : -4b = -\frac{3a}{b}$
8. $-14a : -4b = \frac{7a}{2b}$

9. $abc : a = bc$ 10. $abc : ad = \frac{bc}{d}$

11. $8fmn : -2fgm = -\frac{4n}{g}$

12. $-12abcde : -8acd = \frac{3be}{2}$

13. $6abde : -2bf = -\frac{3ade}{f}$

14. $27a^3b^2cfg : -18abcghk = -\frac{3a^2bf}{2hk}$

15. $35abfgm : 5a^2bf^2gmn = \frac{7}{afn}$

16. $\frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$

17. $\frac{3afx}{bc} : \frac{2fx^2}{5cde} = \frac{15ade}{2bx}$

18. $3fm : -\frac{3am}{5bg} = -\frac{5bfg}{a}$

19. $\frac{2ay}{5bcx} : 3ac = \frac{2y}{15bc^2x}$

20. $\frac{1}{7fg^2l} : \frac{1}{4fln} = \frac{4n}{7g^2}$

21. $\frac{3}{4}ac : \frac{5}{6}abd = \frac{9c}{10bd}$

(*b*) *Division of Compound Magnitudes.*

1. $\left(3ac-2ade-f+\frac{c}{d}\right) : 2a = \frac{3c}{2} - de - \frac{f}{2a} + \frac{c}{2ad}$

2. $(18acf - 6bdef - 2ad) : 3adf = \frac{6c}{d} - \frac{2be}{a} - \frac{2}{3f}$

3. $(8a^2-6ab+4c+1) : -2a = -4a+3b-\frac{2c}{a} - \frac{1}{2a}$

4. $(12acfg - 4af^2g + 3fg^2h) : 4a^2b^2fg = \frac{3c}{ab^2} - \frac{f}{ab^2} + \frac{3gh}{4a^2b^2}$

5. $\left(\frac{a}{b} + \frac{fd}{2c} - 3ac + 7\right) : \frac{3c}{d} = \frac{ad}{3bc} + \frac{fd^2}{6c^2} - ad + \frac{7d}{3c}$

6. $(ab - ac) : (b - c) = a$

7. $(ac - bc + ad - bd) : (a - b) = c + d$

8. $(4a^2 + 6ab - 4ax + 9bx - 15x^2) : (2a + 3x) = 2a + 3b - 5x$

9. $(14af - 21bf + 7cf + 6ag - 9bg + 3cg) : (7f + 3g) = 2a - 3b + c$

10. $(4x^3 + 4x^2 - 29x + 21) : (2x - 3) = 2x^2 + 5x - 7$

11. $(\frac{5}{2}x^3 - \frac{5}{4}x^2 - 8x + 9) : (\frac{1}{2}x - 1) = 3x^2 + \frac{7}{2}x - 9$

12. $(a^2 + ab + 2ac - 2b^2 + 7bc - 3c^2) : (a + 2b - c) = a - b + 3c$

13. $(12a^2 + 26ab - 36ac + 18ad - 10b^2 + 29bc - 6bd - 21c^2 + 9cd) : (6a - 2b + 3c) = 2a + 5b - 7c + 3d$

14. $(119c^2 - 200cd + 408ce - 113cf - 39d^2 + 72de + 37df - 96ef + 20f^2) : (17c + 3d - 4f) = 7c - 13d + 24e - 5f$

15. $\left(3a^2 - \frac{7ab}{2} - \frac{21ac}{4} - \frac{5b^2}{2} + \frac{83bc}{8} - \frac{3c^2}{2}\right) : \left(3a - 5b + \frac{3c}{4}\right) = a + \frac{b}{2} - 2c$

16. $\left(2f^2 - \frac{55fh}{12} + \frac{29fx}{9} + \frac{21h^2}{8} - \frac{15hx}{4} + \frac{x^2}{3}\right) : \left(\frac{2f}{3} - \frac{3h}{4} + x\right) = 3f - \frac{7h}{2} + \frac{x}{3}$

17. $(30a^2b - 6a^2c + 75ab^2 - 15abc) : (15ab - 3ac) = 2a + 5b$

18. $(36a^2b - 63ab^2 + 20b^3) : (12ab - 5b^2) = 3a - 4b$

19. $(72x^4 - 78x^3y - 10x^2y^2 + 17xy^3 + 3y^4) : (6x^2 - 4xy - y^2) = 12x^2 - 5xy - 3y^2$

20. $(\frac{1}{3}x^4 - \frac{11}{12}x^3 + \frac{41}{8}x^2 - \frac{25}{4}x + 6 : (\frac{2}{3}x^2 - \frac{5}{6}x + 1) = \frac{1}{2}x^2 - \frac{3}{4}x + 6$

21. $\left(\frac{15a^2}{2} - \frac{113ab}{6} + 9ac + 2b^2 - bc\right) : \left(\frac{5a}{4} - 3b + \frac{3c}{2}\right) = 6a - \frac{2b}{3}$

22. $\left(-\frac{5x^2}{9} + \frac{11xy}{3} - \frac{10xz}{3} + \frac{15y^2}{4} + 25yz\right) : (-\frac{2}{3}x + 5y) = \frac{5x}{6} + \frac{3y}{4} + 5z$

23. $(18a^2 + 33ab + 42ac - 12ad - 30b^2 + 124bc + 8bd - 16c^2 - 32cd) : (6a + 15b - 2c - 4d) = 3a - 2b + 8c$

24. $(108a^2 - 33ac - 9ad - 9ae - 24ab + 10bc + 2bd + 2be - 5c^2 - cd - ce) : (12a - 5c - d - e) = 9a - 2b + c$

25. $\left(\frac{5}{2}xy + \frac{85}{9}y^2 - \frac{225}{54}yz - \frac{10}{9}yu - \frac{3}{4}xz + \frac{7}{18}z^2 + \frac{uz}{3}\right) : (\frac{3}{2}x + \frac{17}{3}y - \frac{7}{3}z - \frac{2}{3}u) = \frac{5}{3}y - \frac{1}{2}z$

26. $(-75a^2bfx^2 + 65a^2cxy + 60abxy^2 - 65axz + 180abfx^2y - 156acxy^2 - 144bxy^3 + 156xyz) : (15abfx - 12by^2 - 13acy + 13z) = -5ax + 12xy$

27. $\left(\frac{a^2}{bc} - \frac{2a}{d} + \frac{ac}{be} + \frac{bc}{d^2} - \frac{c^2}{de}\right) : \left(\frac{a}{b} - \frac{c}{d}\right) = \frac{a}{c} - \frac{b}{d} + \frac{c}{e}$

28. $\left(\frac{5a^2x}{b} - \frac{5aby}{c} + 5ad - ax + \frac{b^2y}{c} - bd\right) :$
$\left(\frac{ax}{b} - \frac{by}{c} + d\right) = 5a - b$

29. $\left(\frac{a^2x^3}{bd} + \frac{abx^2}{c^2d} - \frac{acx^2}{d^2} - \frac{b^2x}{cd^2} + \frac{a^2x}{bc} - \frac{a}{d}\right) :$
$\left(\frac{ax}{c} - \frac{b}{d}\right) = \frac{acx^2}{bd} + \frac{bx}{cd} + \frac{a}{b}$

30. $(a^2 - b^2) : (a - b) = a + b$

31. $(a^4 - 9a^2b^2 - 6abc^2 - c^4) : (a^2 - 3ab - c^2)$
$= a^2 + 3ab + c^2$

32. $(a^4 - b^4) : (a - b) = a^3 + a^2b + ab^2 + b^3$

33. $(32a^5 + b^5) : (2a + b) = 16a^4 - 8a^3b + 4a^2b^2$
$- 2ab^3 + b^4$

34. $\left(\frac{9a^2b^2}{4c^2} - \frac{25f^2m^2}{g^2} + \frac{70dfm}{g} - 49d^2\right) : \left(\frac{3ab}{2c} + \frac{5fm}{g} - 7d\right)$
$= \frac{3ab}{2c} - \frac{5fm}{g} + 7d$

(c) *Partial Division, being examples in which the Divisor is not an aliquot part of the Dividend.*

1. $1 : (1-b) = 1 + \frac{b}{1-b}$
$= 1 + b + \frac{b^2}{1-b}$
$= 1 + b + b^2 + \frac{b^3}{1-b}$
$= 1 + b + b^2 + b^3 + \frac{b^4}{1-b}$
$= 1 + b + b^2 + b^3 + b^4 + \ldots.$

2. $1 : (1 + b) = 1 - \dfrac{b}{1 + b}$

$= 1 - b + \dfrac{b^2}{1 + b}$

$= 1 - b + b^2 - \dfrac{b^3}{1 + b}$

$= 1 - b + b^2 - b^3 + \dfrac{b^4}{1 + b}$

$= 1 - b + b^2 - b^3 + b^4 - \ldots.$

3. $c : (a-b) = \dfrac{c}{a} + \dfrac{bc}{a(a-b)}$

$= \dfrac{c}{a} + \dfrac{bc}{a^2} + \dfrac{b^2c}{a^2(a-b)}$

$= \dfrac{c}{a} + \dfrac{bc}{a^2} + \dfrac{b^2c}{a^3} + \dfrac{b^3c}{a^3(a-b)}$

$= \dfrac{c}{a} + \dfrac{bc}{a^2} + \dfrac{b^2c}{a^3} + \dfrac{b^3c}{a^4} + \ldots.$

4. $c : (a+b) = \dfrac{c}{a} - \dfrac{bc}{a(a+b)}$

$= \dfrac{c}{a} - \dfrac{bc}{a^2} + \dfrac{b^2c}{a^2(a+b)}$

$= \dfrac{c}{a} - \dfrac{bc}{a^2} + \dfrac{b^2c}{a^3} - \dfrac{b^3c}{a^3(a+b)}$

$= \dfrac{c}{a} - \dfrac{bc}{a^2} + \dfrac{b^2c}{a^3} - \dfrac{b^3c}{a^4} + \ldots.$

5. $(1+x) : (1-x) = 1 + \dfrac{2x}{1-x}$

$= 1 + 2x + \dfrac{2x^2}{1-x}$

$= 1 + 2x + 2x^2 + \dfrac{2x^3}{1-x}$

$= 1 + 2x + 2x^2 + 2x^3 + \ldots$

III. THE ARITHMETIC OF POWERS.

WHAT is meant by the Powers of a Magnitude? What by the Exponent of a Power? What is the Base or Cardinal number of a Power? Is the value of a Power altered when its Base and Exponent are interchanged? How are Powers having the same Base, multiplied and divided? How, when they have different Bases? Should we in Division meet with Powers whose Exponents are 0 or negative, what meaning have such Powers? Are the former rules of Multiplication and Division applicable to such Powers? The Exponent 1 may be either omitted or expressed. In the Addition and Subtraction of Powers is there any abbreviation? What is the consequence, if in the Products and Quotients of Powers the like abbreviation is found? What change takes place in the Exponent of a Power, when that Power is removed from the Numerator to the Denominator of a Fraction, and *vice versa*?

ADDITION AND SUBTRACTION.

1. $ax^n + bx^n + cx^n + dx^n = (a + b + c + d)x^n$
2. $ax^n + bx^n - cx^n - dx^n = (a + b - c - d)x^n$
3. $10a^4 + 3a^4 + 6a^4 - a^4 - 5a^4 = 13a^4$
4. $3a^{-7} + 10a^{-7} - 5a^{-7} + a^2b = 8a^{-7} + a^2b$
5. $6^4 + 2\cdot8^3 + 3^2 - 19\cdot6^4 + 5\cdot8^3 = 7\cdot8^3 - 18\cdot6^4 + 3^2$
6. $16a^4b^3c^5 - 6a^4b^3c^5 + 7a^4b^3c^5 = 17a^4b^3c^5$
7. $\frac{5a^3}{b^4} - \frac{7a^3}{b^4} + \frac{11a^3}{b^4} + a^4 = \frac{9a^3}{b^4} + a^4$
8. $a^nb^m - 9a^m + 5a^nb^m + 6a^m + 10a^nb^m = 16a^nb^m - 3a^m$

9. $5a^{-3}b^2 + 7ab^2c - 3a^mb^{-3} - 12ab^2c + 6a^{-3}b^2 - 9^3b^3 + b^{-x} - 8a^mb^{-5} - 3b^{-x} = 11a^{-3}b^2 - 5ab^2c - 11a^mb^{-5} - 9a^3b^3 - 2b^{-x}$

10. $3\cdot2^{-7} + 5^6 - 8\cdot2^{-7} + 3a^nb^{-m} - 13\cdot5^6 + 4a\cdot2^{-7} + ca^nb^{-m} = (4a - 5)2^{-7} - 12\cdot5^6 + (c + 3)a^nb^{-m}$

11. Added.
$$\begin{array}{l} 5a^4b + 3a^{-2}b^2c - 7ab \\ -6a^4b + 2a^{-2}b^2c + 17ab \\ 9a^4b - 8a^{-2}b^2c - 10ab \\ \hline 8a^4b - 3a^{-2}b^2c \end{array}$$

12. Added.
$$\begin{array}{l} 5a^mb^p + 3a^{-3}b^{m-1} - \dfrac{3a^3}{x^p} \\ -3ca^mb^p + 4g^2a^{-3}b^{m-1} - a + \dfrac{10a^3}{x^p} \\ a^mb^p + a + 3a^2b^2 - 2g^2a^{-3}b^{m-1} \\ \hline (6-3c)a^mb^p + (2g^2 + 3)a^{-3}b^{m-1} + \dfrac{7a^3}{x^p} + 3a^2b^2 \end{array}$$

13. Added.
$$\begin{array}{l} 9a^{-3}b^{-2}c^4 - \dfrac{7b}{a^3} \\ \dfrac{18b}{a^3} - 5a^nb^m + c^x - 3\cdot2^5 \\ 3a^nb^m - ha^{-3}b^{-2}c^4 + 3c^x - 5\cdot2^5 \\ \hline (9 - h)a^{-3}b^{-2}c^4 - 2a^nb^m + \dfrac{11b}{a^3} + 4c^x - 8\cdot2^5 \end{array}$$

14. Subtracted.
$$\begin{array}{l} 9a^mx^2 - 13 + 20ab^3x - 4b^mcx^2 \\ 3b^mcx^2 + 9a^mx^2 - 6 + 3ab^3x \\ \hline 17ab^3x - 7b^mcx^2 - 7 \end{array}$$

15. Subtracted.
$$\begin{array}{l} 5a^4 - 7a^3b^2 - 3c^{-1}d^2 + 7d \\ 3a^4 - 15a^3b^2 - 7c^{-1}d^2 - 3a^2 \\ \hline 2a^4 + 8a^3b^2 + 4c^{-1}d^2 + 7d + 3a^2 \end{array}$$

MULTIPLICATION.

(*a*) *Multiplication of Simple Magnitudes.*

1. $a^m \times a^n = a^{m+n}$
2. $a^{-m} \times a^n = a^{-m+n} = a^{n-m}$
3. $a^m \times a^{-n} = a^{m-n}$
4. $a^{-m} \times a^{-n} = a^{-m-n} = a^{-(m+n)}$
5. $5a^3 \times a^7 \times 7a^5 \times 3a^6 = 105a^{21}$
6. $11a^{-2} \times 2a^{-5} \times 4a^6 \times 9a^7 = 792a^6$
7. $2a^{-3} \times 7a^{-9} \times -3a^6 = -42a^{-6} = -\frac{42}{a^6}$
8. $-a^{-5}b \times -a^{-7}d \times 10a = 10a^{-11}bd = \frac{10bd}{a^{11}}$
9. $3 \cdot 7^{-9} \times 7^{-2} \times 4 \cdot 7^3 = 12 \cdot 7^{-3} = \frac{12}{343}$
10. $-a^{p-q} \times -3a^{3q-2}f \times 5a^{p+7}cx = 15a^{2p+2q+5}fcx$
11. $5a^3b^{-4} \times 10a^2b^5c \times -3a^7 = -150a^{12}bc$
12. $-7a^{-1}b^4c^{-5} \times 3a^2b^{-5}c = -21ab^{-1}c^{-4} = -\frac{21a}{bc^4}$
13. $5a^3b^4 \times a^2b^3 \times 4acb^{-3} = 20a^6b^9c$
14. $h^4l^{11}x \times h^{-7}l^{-9} \times 3h^{-2}l^{-6}x^3 = 3h^{-5}l^{-4}x^4 = \frac{3x^4}{h^5l^4}$
15. $-13a^{-1}c^{-3} \times -4a^{-3}b^{-6}c^2 = 52a^{-4}b^{-6}c^{-1} = \frac{52}{a^4b^6c}$
16. $-\frac{5}{2}a^{-2}b^3c^{-m}d^{-1} \times \frac{2}{3}a^2b^{-3}c^{-2}d^4 = -\frac{5}{3}c^{-m-2}d^3 = -\frac{5b^3}{3c^{m+2}}$
17. $a^{-m}b^pc^q \times a^nb^{-r}c^s \times a^{n+m}b = a^{2n}b^{p-r+1}c^{q+s}$
18. $f^xg^{m-1}hd \times 2f^{m+3}h^{2-m} \times 4g^{2m+1}h^{m+2}d^3 = 8f^{x+m+3}g^{3m}h^5d^4$

19. $(a+y)^{-3}h^5l^4 \times (a+y)^{m+3}l^{-4}m \times (a+y) = mh^5(a+y)^{m+1}$

20. $\frac{18a^{-5}b^3}{7c^{-2}d^{-6}} \times \frac{4a^6b^{-5}}{9c^3d^9} = \frac{72ab^{-2}}{63cd^3} = \frac{8a}{7cd^3b^2}$

21. $\frac{6a^4b^5c^{-7}}{11f^3dg^{-4}} \times \frac{3a^{-2}b^4c^{-1}}{5g^2f^6} = \frac{18a^2b^9c^{-8}}{55f^9dg^{-2}} = \frac{18a^2b^9g^2}{55f^9dc^8}$

22. $\frac{1}{3a^{-3}b^{-m}c} \times \frac{1}{4a^{-p}b} = \frac{1}{12a^{-p-3}b^{-m+1}c} = \frac{a^{p+3}b^{m-1}}{12c}$

23. $\frac{2a^{-m-3}b^{m+2}}{3x^{-5}y^{-n}z^p} \times \frac{6a^{-1}b^{-m}}{x^{-p}y^3} = \frac{12a^{-m-4}b^2}{3x^{-p-5}y^{-n+3}z^p} = \frac{4b^2x^{p+5}y^{n-3}}{a^{m+4}z^p}$

24. $\frac{3a^{-5}bc^3f^4}{(a+b)^{-m}(c^2+x^2)} \times \frac{a^2b^{-5}c^{-2}f^{-4}}{(a+b)^{m-2}(c^2+x^2)^{-n+5}} = \frac{3a^{-3}b^{-3}c}{(a+b)^{-2}(c^2+x^2)^{-n+5}} = \frac{3c(a+b)^2(c^2+x^2)^{n-5}}{a^3b^3}$.

(*b*) *Multiplication of Compound Magnitudes.*

1. $(a^2 - 3ab - 5b^2) \times 4a^2b = 4a^4b - 12a^3b^2 - 20a^2b^3$

2. $(2a^3b^5 - 5a^2c^6 + 9a^3b^2c^3) \times 3a^2bc^2 = 6a^3b^6c^2 - 15a^4bc^8 + 27a^5b^3c^5$

3. $(7h^{-5}l + \frac{2l^3}{h^4} - 3ah^{-3}l^2 + 7) \times -8h^4l^{-5}$
$= -56h^{-1}l^{-4} - 16l^{-2} + 24ahl^{-3} - 56h^4l^{-5}$
$= \frac{24ah}{l^3} - \frac{56}{hl^4} - \frac{56h^4}{l^5} - \frac{16}{l^2}$

4. $\left(a^3b^{-4} - cb^{-5}d^3f + \frac{3c^m}{k^3}\right) \times 2bc^{-2}d = 2a^3b^{-3}c^{-2}d - 2b^{-4}c^{-1}d^4f + \frac{6bc^{m-2}d}{k^3} = \frac{2a^3d}{b^3c^2} - \frac{2d^4f}{b^4c} + \frac{6bc^{m-2}d}{k^3}$

5. $(a^{m-1}b^{2m+3} - 6a^{3-m}b^p + ab^{-m}) \times a^{3m+2}b^{m-1} = a^{4m+1}b^{3m+2} - 6a^{2m+5}b^{p+m-1} + \frac{a^{3m+3}}{b}$

6. $(a + b) \times (a + b) = (a + b)^2 = a^2 + 2ab + b^2$

7. $(a - b) \times (a - b) = (a - b)^2 = a^2 - 2ab + b^2$

8. $(a + b) \times (a - b) = a^2 - b^2$

9. $(a^4 - 2b^3) \times (a - b) = a^5 - 2ab^3 - a^4b + 2b^4$

10. $(x^2 - 3x - 7) \times (x - 2) = x^3 - 5x^2 - x + 14$

11. $(3k^2 - 5kl + 2l^2) \times (k^2 - 7kl) = 3k^4 - 26k^3l + 37k^2l^2 - 14kl^3$

12. $(6f^2 - 17fl + 3l^2) \times (f^5 + 4f^4l) = 6f^7 + 7f^6l - 65f^5l^2 + 12f^4l^3$

13. $(4a^2 - 16ax + 3x^2) \times (5a^3 - 2a^2x) = 20a^5 - 88a^4x + 47a^3x^2 - 6a^2x^3$

14. $(a^2 + a^4 + a^6) \times (a^2 - 1) = a^8 - a^2$

15. $(a^4 - 2a^3b + 4a^2b^2 - 8ab^3 + 16b^4) \times (a + 2b) = a^5 + 32b^5$

16. $(2a^4x^2 - 3b^4y^2) \times (2a^4x^2 + 3b^4y^2) = 4a^8x^4 - 9b^8y^4$

17. $(7a^3 - 5a^2b + 6ab^2 - 2b^3) \times (3a^4 - 4a^3b + 16a^2b^2) = 21a^7 - 43a^6b + 150a^5b^2 - 110a^4b^3 + 104a^3b^4 - 32a^2b^5$

18. $(\frac{5}{2}x^2 + 3ax - \frac{7}{3}a^2) \times (2x^2 - ax - \frac{1}{2}a^2) = 5x^4 + \frac{7}{2}ax^3 - \frac{107}{12}a^2x^2 + \frac{5}{6}a^3x + \frac{7}{6}a^4$

19. $(a^6 - 3a^4b^2 + 5a^2b^4) \times (7a^4 - 4a^2b^2 + b^4) = 7a^{10} - 25a^8b^2 + 48a^6b^4 - 23a^4b^6 + 5a^2b^8$

20. $(a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5) \times (a^3 - 3a^2b + 3ab^2 - b^3) = a^8 - 8a^7b + 28a^6b^2 - 56a^5b^3 + 70a^4b^4 - 56a^3b^5 + 28a^2b^6 - 8ab^7 + b^8$

21. $(a^2 + az + z^2) \times (a^2 - az + z^2) = a^4 + a^2z^2 + z^4$

22. $(15a^{-6}b^2 - 7a^{-5}b^4 + 6a^{-4}b^6) \times (8a^{-2}b^2 - 3a^{-1}b^4) = 120a^{-8}b^4 - 101a^{-7}b^6 + 69a^{-6}b^8 - 18a^{-5}b^{10}$

23. $(13a^{-5}b + 10a^{-2}b^2 - 4ab^3) \times (6a^{-3}b^2 - 18b^3 - 7a^3b^4) = 78a^{-8}b^3 - 174a^{-5}b^4 - 295a^{-2}b^5 + 2ab^6 + 28a^4b^7$

24. $(3x^{-2}y^{-7} - 2x^2y^{-5} + 8x^6y^{-3}) \times (2x^{-3}y^{-5} + 6xy^{-3} + 12x^5y^{-1} = 6x^{-5}y^{-12} + 14x^{-1}y^{-10} + 40x^3y^{-8} + 24x^7y^{-6} + 96x^{11}y^{-4}$

25. $(5a^3b^3c^2 - 6a^4b^2c^5 + 7a^8b^5c^6) \times (2a^3b^3c^2 + 3a^4b^2c^5 - 6a^7b^4c^3) = 10a^6b^6c^4 + 3a^7b^5c^7 + 14a^{11}b^8c^8 - 18a^8b^4c^{10} + 21a^{12}b^7c^{11} - 30a^{10}b^7c^5 + 36a^{11}b^6c^8 - 42a^{15}b^9c^9$

26. $(14a^5c^2 - 6a^2bc^2 + c^3) \times (14a^5c^2 + 6a^2bc^2 - c^3) = 196a^{10}c^4 - 36a^4b^2c^2 + 12a^2bc^5 - c^6$

27. $\left(\frac{a^2}{b^3} + \frac{2c^3d^4}{b^5} - \frac{7c^2}{2a^4b^3}\right) \times \left(\frac{a^2}{b^3} - \frac{2c^3d^4}{b^5} + \frac{7c^2}{2a^4b^3}\right) = \frac{a^4}{b^6} - \frac{4c^6d^8}{b^{10}} + \frac{14c^5d^4}{a^4b^8} - \frac{49c^4}{4a^8b^6}$

28. $(a^m + b^p - 2c^n) \times (2a^m - 3b) = 2a^{2m} + 2a^mb^p - 4a^mc^n - 3a^mb - 3b^{p+1} + 6bc^n$

29. $(2a^{3-2m}b^{n+3} + 3a^{m+1}b^{n+2} + c^p) \times (a^{m-1}b^{1-2m} - ca^p) = 2a^{2-m}b^{n-2m+4} + 3a^{2m}b^{n-2m+3} + a^{m-1}b^{1-2m}c^p - 2ca^{p-2m+3}b^{n+3} - 3ca^{p+m+1}b^{n+2} - a^pc^{p+1}$

30. $(x^{-3p} + 3a^mx^{-2p} - 10a^{2m}x^{-p}) \times (a^2x^q + 5a^{m+2}x^{q+p} - 2a^{2m+2}x^{q+2p}) = a^2x^{q-3p} + 8a^{m+2}x^{q-2p} + 3a^{2m+2}x^{q-p} - 56a^{3m+2}x^q + 20a^{4m+2}x^{q+p}$

31. $(3a^{4-3m}bc^{m-2} + 17a^{-2}b^{m+1}) \times (3a^{6m-2}b^{2m}c^{3-4m} - 8)$
$= 9a^{3m+2}\ b^{2m+1}\ c^{1-3m} + 51a^{6m-5}\ b^{3m+1}\ c^{3-4m}$
$- 24a^{4-3m}\,bc^{m-2} - 136a^{-3}\,b^{m+1}.$

The Forms 6, 7, 8 (b) are important propositions; how are they expressed in words?

(3.) DIVISION.

(a) *Division of Simple Magnitudes.*

1. $a^m : a^n = a^{m-n}$
2. $a^m : a^{-n} = a^{m+n}$
3. $a^{-m} : a^n = a^{-m-n} = a^{-(m+n)}$
4. $a^{-m} : a^{-n} = a^{n-m}$
5. $8a^{10} : 2a^4 = 4a^6$
6. $\frac{7}{3}a^3 : \frac{2}{5}a^7 = \frac{35}{6}a^{-4} = \frac{35}{6a^4}$
7. $\frac{14}{5}a^{-8} : -3a = -\frac{14}{15}a^{-9} = -\frac{14}{15a^9}$
8. $ca^{18} : da^{-6} = \frac{ca^{24}}{d}$
9. $6(a+b)^{-9} : 4(a+b)^{-5} = \frac{3}{2}(a+b)^{-4} = \frac{3}{2(a+b)^4}$
10. $\frac{5}{3}a^{-7}b^3c : \frac{1}{2}a^{-9}b^{-5}c^3f = \frac{10a^2b^8f}{3c^2}$
11. $(a+x)^2\ (a+y)^{-3} : (a+x)^{-4}\ (a+y)^{-7} = (a+x)^6\ (a+y)^4$
12. $-3a^mb^n : -4a^pb^qc^r = \frac{3a^{m-p}b^{n-q}}{4c^r}$

13. $-\frac{5c^2a^{-m}b^n}{8} : 3cd^5a^pb^{-q} = -\frac{5cb^{n+q}}{24d^5a^{m+p}}$

14. $\frac{3a^3d}{2b^5} : \frac{b^3}{4a^2c^7} = \frac{6a^5c^7d}{b^8}$

15. $\frac{2c^3(1+z^2)^2}{d^7z^9} : \frac{5c^8f^3(1+z^2)^{-6}}{2d^9z^5} = \frac{4d^2(1+z^2)^8}{5c^5f^3z^4}$

16. $\frac{2x^{3n-5m}y^{2n-3}}{7a^mb^3c} : \frac{5x^{1-5m}}{3ab^{n-1}y^5} = \frac{3b^{n-4}y^{2n+2}x^{3n-1}}{14a^{m-1}c}$

(*b*) *Division of Compound Magnitudes.*

1. $(6a^3b^2-10a^2f+7a^4bx) : 2a^2 = 3ab^2-5f+\frac{7}{2}a^2bx$

2. $(\frac{3}{4}a^2x^5-\frac{7}{3}ax^3+3ab^2x) : \frac{2}{3}a^2x^3 = \frac{9}{8}x^2 - \frac{7}{6a} + \frac{9b^2}{2ax^2}$

3. $\left(\frac{3a^2b^6}{4} - \frac{5ac}{b^7} + 2a^3c^2 - \frac{2a^2c^2}{5b(a+y)^2}\right) : -\frac{2a^5b^2}{3c}$

$= -\frac{9b^4c}{8a^3} + \frac{15c^2}{2a^4b^9} - \frac{3c^3}{a^2b^2} + \frac{3c^3}{5a^3b^3(a+y)^2}$

4. $(bc^3 - c^3x) : (b - x) = c^3$

5. $(a^2 + 2ab + b^2) : (a + b) = a + b$

6. $(a^3 + a^2b - ab^2 - b^3) : (a-b) = a^2 + 2ab + b^2$

7. $(3a^5 + 16a^4b - 33a^3b^2 + 14a^2b^3) : (a^2 + 7ab)$
$= 3a^3 - 5a^2b + 2ab^2$

8. $(a^7 - 6a^6b^3 + 14a^5b^6 - 12a^4b^9) : (a^3 - 2a^2b^3)$
$= a^4 - 4a^3b^3 + 6a^2b^6$

9. $(a^4 - 2a^2b^2 + b^4) : (a^2 - b^2) = a^2 - b^2$

10. $[a^2bx^8-(a^3b-a^5)x^7-8a^6x^6+7a^7x^5] : (a^2x^2 - a^3x)$
$= bx^6 + a^3x^5 - 7a^4x^4$

11. $(-a^2b^4+15a^{11}b^5-48a^{14}b^6-20a^{17}b^7) : (10a^9b^2-a^6b)$
$= a^2b^3 - 5a^5b^4 - 2a^8b^5$

12. $(a^3-16z^3) : (a^2 - 2z^2) = a^6+2a^4z^2 + 4a^2z^4 + 8z^6$

13. $(2a^4-13a^3b+31a^2b^2-38ab^3+24b^4) : (2a^2-3ab+4b^2)$
$= a^2 - 5ab + 6b^2$

14. $(4c^4 - 9b^2c^2 + 6b^3c - b^4) : (2c^2 - 3bc + b^2)$
$= 2c^2 + 3bc - b^2$

15. $(\frac{3}{4}x^5 - 4x^4 + \frac{77}{8}x^3 - \frac{43}{4}x^2 - \frac{33}{4}x + 27) : (\frac{1}{2}x^2 - x + 3)$
$= \frac{3}{2}x^3 - 5x^2 + \frac{1}{4}x + 9$

16. $(-1 + a^3n^3) : (-1 + an) = 1 + an + a^2n^2$

17. $(3a^4b^{12} - 8a^7b^8 - \frac{51}{2}a^{10}b^6 + \frac{5}{4}a^{10}b^4 + \frac{17}{4}a^{13}b^2)$
$: (\frac{3}{2}a^3b^5 - \frac{1}{4}a^6b) = 2ab^7 - 5a^4b^3 - 17a^7b$

18. $(5a^5b^3c^5 - 22a^4b^3c^6 + 5a^3b^3c^7 + 12a^2b^3c^8 - 7a^2b^2c^8$
$+ 28ab^2c^9) : (a^2bc^2 - 4abc^3) = 5a^3b^2c^3 - 2a^2b^2c^4$
$- 3ab^2c^5 - 7bc^6$

19. $\left(\frac{a^7b^2}{5} - \frac{47a^6b^3}{40} + \frac{9a^5b^4}{2} - 12a^4b^5\right) : \left(\frac{2a^3b^2}{5} - \frac{3a^2b^3}{4}\right.$
$\left. + 6ab^4\right) = \frac{1}{2}a^4 - 2a^3b$

20. $(-2a^{-8}x^5 + 17a^{-4}x^6 - 5x^7 - 24a^4x^8) : (2a^{-3}x^3 - 3ax^4)$
$= -a^{-5}x^2 + 7a^{-1}x^3 + 8a^3x^4$

21. $\left(\frac{a^3c}{b^5} + \frac{a^4c}{b^4} - \frac{7a^5c}{b^3} - \frac{3a^6c}{b^2} + \frac{a^2c^3}{b^2} - \frac{2a^3c^3}{b} - a^4c^3\right) :$
$\left(\frac{a}{b^3} + \frac{3a^2}{b^2} + c^2\right) = \frac{a^2c}{b^2} - \frac{2a^3c}{b} - a^4c$

22. $(a^3d^3 - 3a^2cd^3 + 3ac^2d^3 - c^3d^3 + a^2c^2d^2 - ac^3d^2) :$
$(a^2d^2 - 2acd^2 + c^2d^2 + ac^2d) = ad - cd$

23. $(a^6 + 2a^3z^3 + z^6) : (a^2 - az + z^2) = a^4 + a^3z + az^3 + z^4$

24. $(\frac{1}{3} - 6z^2 + 27z^4) : (\frac{1}{3} + 2z + 3z^2) = 1 - 6z + 9z^2$

25. $(a^6 - 16a^3x^3 + 64x^6) : (a^2 - 4ax + 4x^2) = a^4 + 4a^3x$
$+ 12a^2x^2 + 16ax^3 + 16x^4$

26. $(a^{3m-2n}b^{2p}c - a^{2m+n-1}b^{1-p}c^n + a^{-n}b^{-1}c^m + a^{3m-n}b^{3p+2}c^n$
$- a^{2m+2n-1}b^3c^{2n-1} + b^{p+1}c^{m+n-1}) : (a^{-n}b^{-p-1} + bc^{n-1})$
$= a^{3m-n}b^{3p+1}c - a^{2m+2n-1}b^2c^n + b^2c^m$

27. $(a^{m+n}b^n - 4a^{m+n-1}b^{2n} - 27a^{m+n-2}b^{3n} + 42a^{m+n-3}b^{4n})$
$: (a^nb^n - 7a^{n-1}b^{2n}) = a^m + 3a^{m-1}b^n - 6a^{m-2}b^{2n}$

28. $(a^n - b^n) : (a - b) = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots$
$\ldots . + b^{n-1}$

(c) Cases in which the Divisor is not an aliquot part of the Dividend.

1. $\dfrac{a}{1+x} = a - ax + ax^2 - ax^3 + ax^4 - \ldots .$

2. $\dfrac{a}{1-x} = a + ax + ax^2 + ax^3 + ax^4 + \ldots .$

3. $\dfrac{a}{x+1} = \dfrac{a}{x} - \dfrac{a}{x^2} + \dfrac{a}{x^3} - \dfrac{a}{x^4} + \ldots$

4. $\dfrac{a}{x-1} = \dfrac{a}{x} + \dfrac{a}{x^2} + \dfrac{a}{x^3} + \dfrac{a}{x^4} + \ldots .$

5. $\dfrac{a+x}{b+x} = \dfrac{a}{b} - \dfrac{a-b}{b^2}x + \dfrac{a-b}{b^3}x^2 - \dfrac{a-b}{b^4}x^3 + \ldots .$

6. $\dfrac{a-x}{b-x} = \dfrac{a}{b} + \dfrac{a-b}{b^2}x + \dfrac{a-b}{b^3}x^2 + \dfrac{a-b}{b^4}x^3 + \ldots .$

7. $\dfrac{x+a}{x-b} = 1 + \dfrac{a+b}{x} + \dfrac{b(a+b)}{x^2} + \dfrac{b^2(a+b)}{x^3} + \ldots$

(4) POWERS OF POWERS.

1. $\left[\left((a^m)^n\right)^p\right]^q = a^{mnpq}$
2. $\left[\left((a^{-m})^{-n}\right)^p\right]^q = a^{mnpq}$
3. $\left[\left((a^{-m})^{-n}\right)^{-p}\right]^{-q} = a^{mnpq}$
4. $\left[\left((a^m)^{-n}\right)^{-p}\right]^{-q} = a^{-mnpq}$
5. $\left[(a^3bc^2)^5\right]^6 = a^{90}b^{30}c^{60}$
6. $(a^{-2}b^3c^{-5}f^6x^{-1})^{-3} = a^6b^{-9}c^{15}f^{-18}x^3$
7. $(a^mb^{-n}c^pd)^r = a^{mr}b^{-nr}c^{pr}d^r$
8. $(a^{3m-n}f^{2n-1}x^n)^{-3m} = a^{3mn-9mm}f^{3m-6mn}x^{-3mn}$
9. $[a^3(a+b)^2]^m = a^{3m}(a+b)^{2m}$
10. $\left(\dfrac{a^mb^nc^pd^{-q}}{f^ng^{-m}}\right)^{-r} = \dfrac{a^{-mr}b^{-nr}c^{-pr}d^{qr}}{f^{-nr}g^{mr}}$
11. $\left(\dfrac{a^4b^5}{c^3df}\right)^4 = \dfrac{a^{16}b^{20}}{c^{12}d^4f^4}$
12. $\left[\left(\dfrac{a^2b^3}{cd^5}\right)^{-1}\right]^{-2m} = \dfrac{a^{4m}b^{6m}}{c^{2m}d^{10m}} = \left(\dfrac{a^4b^6}{c^2d^{10}}\right)^m$
13. $(-a^2)^5 = -a^{10}$
14. $(-b^{-3})^4 = b^{-12}$
15. $\left[\left((-a)^3\right)^4\right]_5 = a^{60}$
16. $\left[(-a)^{-3}\right]^{-5} = -a^{15}$
17. $\left[(-a)^{-4}\right]^{-6} = a^{24}$
18. $(-a)^{2m} = a^{2m}$
19. $(-a)^{2m+1} = -a^{2m+1}$
20. $\left[\left(-\dfrac{a}{b}\right)^3\right]^{-4} = \dfrac{b^{12}}{a^{12}}$

IV. ON ROOTS.

WHAT is meant by extracting the Root of a Number? What is a Root? Are there numbers whereof the Roots can neither be expressed in Integers, nor in Vulgar or Decimal Fractions? How are such numbers to be treated? And what is the name of their Roots? What do the words Commensurable and Incommensurable denote? Irrational numbers compared with Integers or Vulgar Fractions, are necessarily Commensurables: but are they so when compared with one another? Give examples in which they are not so. In the Addition and Subtraction of Roots what abbreviations are there, and upon what conditions? What is the abbreviation admissible in the Multiplication and Division of Roots? How is the Radical Exponent increased? Since, when a root is considered as unconnected with others, it is desirable to have the Radical Exponent as small as possible, for what purpose is the above required?

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(1) SQUARE AND CUBE ROOTS OF NUMBERS.

(a) Square Roots.

1. $\sqrt{256} = 16$
2. $\sqrt{4096} = 64$
3. $\sqrt{61009} = 247$

$\sqrt{5239521} = 3 \times 763 = 2289$

4. $\sqrt{582169} = 763$
5. $\sqrt{956484} = 978$
6. $\sqrt{57198969} = 7563$
7. $\sqrt{68492176} = 8276$
8. $\sqrt{25836889} = 5085$
9. $\sqrt{236144689} = 15367$
10. $\sqrt{1607448649} = 40093$
11. $\sqrt{780811249} = 27943$
12. $\sqrt{1420913025} = 37695$
13. $\sqrt{285970396644} = 534762$
14. $\sqrt{41605800625} = 203975$
15. $\sqrt{48303584206084} = 6950078$
16. $\sqrt{12088868379025} = 3476905$
17. $\sqrt{5} = 2{\cdot}23606....$
18. $\sqrt{13} = 3{\cdot}60555....$
19. $\sqrt{22} = 4{\cdot}69041....$
20. $\sqrt{96} = 9{\cdot}79795....$
21. $\sqrt{153} = 12{\cdot}36931....$
22. $\sqrt{101} = 10{\cdot}04987....$
23. $\sqrt{7{\cdot}65} = 2{\cdot}76586....$
24. $\sqrt{9{\cdot}6} = 3{\cdot}09838....$
25. $\sqrt{15{\cdot}2379} = 3{\cdot}90357....$
26. $\sqrt{0{\cdot}056} = 0{\cdot}23664....$
27. $\sqrt{0{\cdot}00789} = 0{\cdot}08882....$
28. $\sqrt{0{\cdot}003} = 0{\cdot}05477....$
29. $\sqrt{0{\cdot}014} = 0{\cdot}11832....$
30. $\sqrt{\frac{1}{4}} = \frac{1}{2}$
31. $\sqrt{\frac{9}{16}} = \frac{3}{4}$

32. $\sqrt[3]{\frac{64}{81}} = \frac{4}{9}$
33. $\sqrt[3]{\frac{256}{361}} = \frac{16}{19}$
34. $\sqrt[3]{\frac{7}{4}} = 1{\cdot}32287....$
35. $\sqrt[3]{\frac{14}{9}} = 1{\cdot}24721....$
36. $\sqrt[3]{11\frac{11}{16}} = 3{\cdot}41869....$
37. $\sqrt[3]{7\frac{15}{56}} = 2{\cdot}71313....$
38. $\sqrt[3]{8\frac{15}{49}} = 2{\cdot}88203....$
39. $\sqrt[3]{\frac{5}{3}} = 1{\cdot}29099....$
40. $\sqrt[3]{\frac{7}{8}} = 0{\cdot}93541....$
41. $\sqrt[3]{\frac{5}{12}} = 0{\cdot}64549....$
42. $\sqrt[3]{\frac{1}{17}} = 0{\cdot}24253....$
43. $\sqrt[3]{10\frac{5}{10}} = 3{\cdot}20936....$

(b) Cube Roots.

1. $\sqrt[3]{12167} = 23$
2. $\sqrt[3]{884736} = 96$
3. $\sqrt[3]{405224} = 74$
4. $\sqrt[3]{2460375} = 135$
5. $\sqrt[3]{11089567} = 223$
6. $\sqrt[3]{1191016} = 106$
7. $\sqrt[3]{17173512} = 258$
8. $\sqrt[3]{49836032} = 368$
9. $\sqrt[3]{40353607} = 343$
10. $\sqrt[3]{64481201} = 401$
11. $\sqrt[3]{8000000} = 200$

12. $\sqrt[3]{74088000} = 420$
13. $\sqrt[3]{318611987} = 683$
14. $\sqrt[3]{340068392} = 698$
15. $\sqrt[3]{6372783864} = 1854$
16. $\sqrt[3]{7256313856} = 1936$
17. $\sqrt[3]{111980168000} = 4820$
18. $\sqrt[3]{115145914625} = 4865$
19. $\sqrt[3]{18970074963} = 2667$ $\sqrt[3]{15176059704} = 2 \times 2667 = 5334$
20. $\sqrt[3]{113028882875} = 4835$ $\sqrt[3]{[illegible]} = 11 \times 2667 = 29337$ $\sqrt[3]{25249169775753} =$
21. $\sqrt[3]{8108486729} = 2009$
22. $\sqrt[3]{12} = 2{\cdot}28942....$
23. $\sqrt[3]{82} = 4{\cdot}34448....$
24. $\sqrt[3]{267} = 6{\cdot}43927....$
25. $\sqrt[3]{551} = 8{\cdot}19817....$
26. $\sqrt[3]{687} = 8{\cdot}82373....$
27. $\sqrt[3]{5{\cdot}8} = 1{\cdot}79670....$ 1.796701801
28. $\sqrt[3]{102{\cdot}875} = 4{\cdot}68565....$
29. $\sqrt[3]{28{\cdot}25} = 3{\cdot}04559....$
30. $\sqrt[3]{\sqrt[5]{48230{\cdot}605376}} = 38{\cdot}76$
31. $\sqrt[3]{\frac{1}{8}} = \frac{1}{2}$
32. $\sqrt[3]{\frac{27}{64}} = \frac{3}{4}$
33. $\sqrt[3]{\frac{343}{512}} = \frac{7}{8}$

$\sqrt[3]{5.8} = 1.7967 + \text{rem}^r \cdot 1722993?$

34. $\sqrt[3]{\frac{729}{125}} = \frac{9}{5}$

35. $\sqrt[3]{465\frac{31}{64}} = 7\frac{3}{4}$

36. $\sqrt[3]{52034\frac{10}{27}} = 37\frac{1}{3}$

37. $\sqrt[3]{\frac{2}{3}} = 0{\cdot}87358....$

38. $\sqrt[3]{\frac{5}{6}} = 0{\cdot}94103....$

39. $\sqrt[3]{\frac{5}{14}} = 0{\cdot}70949....$

40. $\sqrt[3]{3\frac{4}{5}} = 1{\cdot}56049....$

41. $\sqrt[3]{15\frac{2}{3}} = 2{\cdot}50222....$

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(2) ROOTS OF LITERAL EXPRESSIONS

(a) Roots of Simple Literal Expressions.

1. $\sqrt[m]{a^{mn}} = a^{\frac{mn}{m}} \quad a^n$

2. $\sqrt[m]{a^{-mn}} = a^{-n}$

3. $\sqrt[m]{a^{mn}b^{mp}c^{-mq}d^{-mr}} = a^n b^p c^{-q} d^{-r}$

4. $\sqrt[m]{\frac{a^{mn}b^{mp}}{c^{mq}d^{mr}f^{ms}}} = \frac{a^n b^p}{c^q d^r f^s}$

5. $\sqrt{9a^4b^2f^{-12}g^{-8n}} = 3a^2bf^{-6}g^{-4n}$

6. $\sqrt{\frac{4}{25}c^6x^{2m+4}y^{4p-8n}} = \frac{2}{5}c^3x^{m+2}y^{2p-n}$

7. $\sqrt[4]{\frac{a^8b^{20}c^4}{(a+f)^4h^{12}z^{16}}} = \frac{a^2b^5c}{(a+f)h^3z^4}$

8. $\sqrt[3]{\frac{27w^9x^{12}(a^2+x^2)^{-6n}}{8b^{-3n}h^9}} = \frac{3b^n w^3 x^4}{2h^3(a^2+x^2)^{2n}}$

9. $\sqrt[5]{\frac{3^{10}a^{-5}(a+b)^5(2+x)^{-10}}{32c^5d^{-15}}} = \frac{9d^3(a+b)}{2ac(2+x)^2}$

10. $\sqrt[9]{\left(2^{36}a^{45}b^{9} \times \frac{(a+b)^{27}}{a^{9}}\right)} = 2^{4}a^{4}b(a+b)^{3}$

11. $\sqrt{\left(\frac{a^{2}b^{2}c^{2}}{d^{4}} \times \frac{1}{a^{6}b^{8}f^{4}}\right)} = \frac{c}{a^{2}b^{3}d^{2}f^{2}}$

(b) *Square Roots of Compound Literal Expressions.*

1. $\sqrt{(a^{2} + 2ab + b^{2})} = a + b$
2. $\sqrt{(a^{2} - 2ab + b^{2})} = a - b$
3. $\sqrt{\left(a^{2} - ab + \frac{b^{2}}{4}\right)} = a - \frac{b}{2}$
4. $\sqrt{(x^{2} + 2x + 1)} = x + 1$
5. $\sqrt{(f^{6} + 6f^{3}x^{4} + 9x^{8})} = f^{3} + 3x^{4}$
6. $\sqrt{\left(\frac{9a^{8}}{4} + 2a^{4}n^{3} + \frac{4n^{6}}{9}\right)} = \frac{3a^{4}}{2} + \frac{2n^{3}}{3}$
7. $\sqrt{(\frac{25}{4}a^{2}b^{2} - \frac{5}{2}abc^{2} + \frac{1}{9}c^{4})} = \frac{5}{2}ab - \frac{1}{3}c^{2}$
8. $\sqrt{(x^{4} - ax^{3} + \frac{1}{4}a^{2}x^{2})} = x^{2} - \frac{1}{2}ax$
9. $\sqrt{(a^{2m} + 2a^{m}x^{n} + x^{2n})} = a^{m} + x^{n}$
10. $\sqrt{(a^{2m} - 4a^{m+n} + 4a^{2n})} = a^{m} - 2a^{n}$
11. $\sqrt{\left(\frac{a^{2}}{b^{2}} - \frac{4a}{3c} + \frac{4b^{2}}{9c^{2}}\right)} = \frac{a}{b} - \frac{2b}{3c}$
12. $\sqrt{(a^{2} + 2ab + 2ac + b^{2} + 2bc + c^{2})} = a + b + c$
13. $\sqrt{\left(9x^{2} - 30ax - 3a^{2}x + 25a^{2} + 5a^{3} + \frac{a^{4}}{4}\right)}$
$= 3x - 5a - \frac{a^{2}}{2}$
14. $\sqrt{(4x^{4} + 8ax^{3} + 4a^{2}x^{2} + 16b^{2}x^{2} + 16ab^{2}x + 16b^{4})}$
$= 2x^{2} + 2ax + 4b^{2}$
15. $\sqrt{(9a^{2} - 6ab + 30ac + 6ad + b^{2} - 10bc - 2bd + 25c^{2} + 10cd + d^{2})} = 3a - b + 5c + d$

16. $\sqrt{(\frac{9}{1} + 6x - 17x^2 - 28x^3 + 49x^4)} = \frac{3}{2} + 2x - 7x^2$

17. $\sqrt{(9x^4 - 3ax^3 + 6bx^3 + \frac{a^2x^2}{4} - abx^2 + b^2x^2)}$

$= 3x^2 - \frac{ax}{2} + bx$

18. $\sqrt{(\frac{4}{9}a^2x^4 - \frac{4}{3}abx^3z + \frac{8}{3}a^2bx^2z^2 + b^2x^2z^2 - 4ab^2xz^3 + 4a^2b^2z^4)} = \frac{2}{3}ax^2 - bxz + 2abz^2$

19. $\sqrt{\frac{a^2-2ab+b^2}{x^4+4ax^2+4a^2}} = \frac{a-b}{x^2+2a}$

20. $\sqrt{\frac{a^2x^2+2ab^2x^3+b^4x^4}{a^{2m}+2a^mx^n+x^{2n}}} = \frac{ax+b^2x^2}{a^m+x^n}$

21. $\sqrt{(a^{2m}x^{2n} + 10ca^{2m-2}x^{2n+1} - 6a^{m+1}x^{n-1} + 25c^2a^{2m-4}x^{2n+2}}$

$- 30ca^{m-1}x^n + \frac{9a^2}{x^2} = a^mx^n + 5ca^{m-2}x^{n+1} - \frac{3a}{x}$

22. $\sqrt{\left(\frac{9a^{2m-2}c^2}{4d^{6p}} - \frac{3a^{m+n-1}b^{2n-1}c}{d^{3p-3}} - \frac{2^8a^{m-1}b^xc}{d^{3p}} + a^{2n}b^{4n-2}d^6\right.}$

$\left. + \frac{2^9}{3}a^nb^{x+2n-1}d^3 + \frac{2^{16}b^{2x}}{9}\right) = \frac{3a^{m-1}c}{2d^{3p}} - a^nb^{2n-1}d^3$

$- \frac{2^8b^x}{3}$

(c) *Cube Roots of Compound Literal Expressions.*

1. $\sqrt[3]{(a^3 + 3a^2b + 3ab^2 + b^3)} = a + b$

2. $\sqrt[3]{(a^3 - 3a^2b + 3ab^2 - b^3)} = a - b$

3. $\sqrt[3]{(x^3 + 6x^2 + 12x + 8)} = x + 2$

4. $\sqrt[3]{(8a^3 - 84a^2x + 294ax^2 - 343x^3)} = 2a - 7x$

5. $\sqrt[3]{(x^6 - 6cx^5 + 12c^2x^4 - 8c^3x^3)} = x^2 - 2cx$

6. $\sqrt[3]{(a^{3m} - 6a^{2m+1}x^n + 12a^{m+2}x^{2n} - 8a^3x^{3n})} = a^m - 2ax^n$

7. $\sqrt[3]{(8 - 12x^{3n-1} + 6x^{6n-2} - x^{9n-3})} = 2 - x^{3n-1}$

8. $\sqrt[3]{\left(\frac{a^3c^3}{b^3}x^6 - \frac{3a^2c}{b}x^5 + \frac{3ab}{c}x^4 - \frac{b^3}{c^3}x^3\right)} = \frac{ac}{b}x^2 - \frac{b}{c}x$

9. $\sqrt[3]{\left(b^3 + \frac{3a^2b^2}{2c^2}x^{-2} + \frac{3a^4b}{4c^4}x^{-4} + \frac{a^6}{8c^6}x^{-6}\right)} = b + \frac{a^2}{2c^2}x^{-2} = b + \frac{a^2}{2c^2x^2}$

10. $\sqrt[3]{(a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3)} = a + b + c$

11. $\sqrt[3]{(27z^6 - 54az^5 + 63a^2z^4 - 44a^3z^3 + 21a^4z^2 - 6a^5z + a^6)} = 3z^2 - 2az + a^2$

12. $\sqrt[3]{\left(\frac{a^3y^3}{b^6c^3} + \frac{3a^2cy^4}{b^4d} - \frac{3a^3y^2}{b^4c^2} + \frac{3ac^5y^5}{b^2d^2} - \frac{6a^2c^2y^3}{b^2d} + \frac{3a^3y}{b^2c} + \frac{c^9y^6}{d^3} - \frac{3ac^6y^4}{d^2} + \frac{3a^2c^3y^2}{d} - a^3\right)} = \frac{ay}{b^2c} + \frac{c^3y^2}{d} - a$

13. $\sqrt[3]{(8x^6 + 48cx^5 + 60c^2x^4 - 80c^3x^3 - 90c^4x^2 + 108c^5x - 27c^6)} = 2x^2 + 4cx - 3c^2$

14. $\sqrt[3]{[(a+b)^{6m}x^3 + 6ca^p(a+b)^{4m}x^2 + 12c^2a^{2p} \times (a+b)^{2m}x + 8c^3a^{3p}]} = (a+b)^{2m}x + 2ca^p$

(*d*) *Square and Cube Roots of incomplete Squares and Cubes.*

1. $\sqrt{(a^2 - x^2)} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \ldots$

2. $\sqrt{(a^2 + x^2)} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \ldots$

3. $\sqrt{(1-x)} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \ldots.$

4. $\sqrt{(1+x)} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \ldots.$

5. $\sqrt[3]{(a^3-x^3)} = a - \frac{x^3}{3a^2} - \frac{x^6}{9a^5} - \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}} - \ldots$

6. $\sqrt[3]{(a^3+x^3)} = a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}} + \ldots.$

7. $\sqrt[3]{(1-x)} = 1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \ldots.$

8. $\sqrt[3]{(1+x)} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \ldots.$

The use of these series in the Extraction of Roots will be seen, if the pupil substitutes for a in the forms (1), (2), (5), (6), the numbers 2, 3, 4, 5, &c. and for x constantly 1; and in the forms (3), (4), (7), (8), if he put for x any of the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c.

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(3) THE ARITHMETIC OF ROOTS.

(a) *Addition and Subtraction.*

1. $b\sqrt[m]{a} + c\sqrt[m]{a} - d\sqrt[m]{a} = (b + c - d)\sqrt[m]{a}$

2. $3\sqrt[6]{5} + 17\sqrt[6]{5} - 12\sqrt[6]{5} - 7\sqrt[6]{5} = \sqrt[6]{5}$

3. $6\sqrt{2} - 5\sqrt{2} + \frac{5}{4}\sqrt{2} - \frac{1}{2}\sqrt{2} = \frac{5}{4}\sqrt{2}$

4. $6\sqrt[4]{\frac{5}{2}} - 2\sqrt[4]{\frac{5}{2}} + a\sqrt[4]{\frac{5}{2}} - \frac{2b}{c}\sqrt[4]{\frac{5}{2}} = \left(4 + a - \frac{2b}{c}\right)\sqrt[4]{\frac{5}{2}}$

5. $5\sqrt[7]{9} - 2\sqrt[5]{14} + \sqrt[3]{2} - 5\sqrt[5]{14} - 2\sqrt[7]{9} = 3\sqrt[7]{9} - 7\sqrt[5]{14} + \sqrt[3]{2}$

6. Added.

$$\left.\begin{array}{l} 10\sqrt{2} + 5\sqrt[7]{8} - 7\sqrt[m]{5} + 2\sqrt[3]{a} \\ 5\sqrt{2} + \sqrt[7]{8} + 4\sqrt[m]{5} - 3\sqrt[3]{a} \\ -3\sqrt{2} - 9\sqrt[7]{8} - 3\sqrt[m]{5} + \sqrt[3]{a} + \sqrt{ab} \end{array}\right\}$$

$$12\sqrt{2} - 3\sqrt[7]{8} - 6\sqrt[m]{5} + \sqrt{ab}$$

7. Added.

$$\left.\begin{array}{l} 13\sqrt[5]{12a^2bc} + 17\sqrt[m]{3} - 5\sqrt[n]{6} \\ 7\sqrt[5]{12a^2bc} + 2\sqrt[n]{6} + 3\sqrt[m]{3} - 2a\sqrt{c} + \frac{1}{2}\sqrt[7]{9a} \\ -20\sqrt[5]{12a^2bc} + 9\sqrt[5]{12a^2bc} + \sqrt{c} - 3\sqrt[7]{9a} \end{array}\right\}$$

$$20\sqrt[m]{3} - 3\sqrt[n]{6} + 9\sqrt[5]{12a^2bc} - (2a-1)\sqrt{c} - \tfrac{5}{2}\sqrt[7]{9a}$$

8. Subtracted.

$$\left.\begin{array}{l} 18\sqrt{7} - 5\sqrt{6} + 10\sqrt[4]{11} - 3\sqrt[3]{13} \\ 6\sqrt{7} - 2\sqrt{6} + \frac{17}{2}\sqrt[4]{11} + 2\sqrt[3]{13} \end{array}\right\}$$

$$12\sqrt{7} - 3\sqrt{6} + \tfrac{3}{2}\sqrt[4]{11} - 5\sqrt[3]{13}$$

9. Subtracted.

$$\left.\begin{array}{l} 16\sqrt[4]{6ab} - \sqrt[5]{9c^3} + 3\sqrt[m]{7a} - \sqrt{10} \\ -8\sqrt[5]{9c^3} - 5\sqrt[m]{7a} + 3\sqrt[4]{6ab} + 2\sqrt[4]{10} \end{array}\right\}$$

$$13\sqrt[4]{6ab} + 7\sqrt[5]{9c^3} + 8\sqrt[m]{7a} - \sqrt{10} - 2\sqrt[4]{10}$$

(*b*) *Abbreviations and Transformations.*

1. $\sqrt{24} + \sqrt{54} - \sqrt{6} = 4\sqrt{6}$
2. $2\sqrt{8} - 7\sqrt{18} + 5\sqrt{72} - \sqrt{50} = 8\sqrt{2}$
3. $\sqrt{12} + 2\sqrt{27} + 3\sqrt{75} - 9\sqrt{48} = -13\sqrt{3}$
4. $8\sqrt{\frac{3}{4}} - \frac{1}{2}\sqrt{12} + 4\sqrt{27} - 2\sqrt{\frac{5}{16}} = \frac{29}{2}\sqrt{3} = \dfrac{29\sqrt{3}}{2}$

5. $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} + \sqrt{\frac{3}{5}} = \frac{26}{15}\sqrt{15}$

6. $7\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{2} - 5\sqrt[3]{128} = 8\sqrt[3]{2}$

7. $\sqrt[3]{81} - 2\sqrt[3]{24} + \sqrt{28} + 2\sqrt{63} = 8\sqrt{7} - \sqrt[3]{3}$

8. $\sqrt[4]{32} + 2\sqrt[3]{40} = 2\sqrt[4]{2} + 4\sqrt[3]{5}$

9. $3\sqrt{5} - 2\sqrt{2} + 3\sqrt{6} = \sqrt{45} - \sqrt{8} + \sqrt{54}$

10. $5\sqrt[3]{7} + 3\sqrt{2} + 2\sqrt[4]{3} = \sqrt[3]{875} + \sqrt{18} + \sqrt[4]{48}$

11. $4\sqrt[5]{\frac{1}{2}} + 3\sqrt[3]{2} - 5\sqrt{\frac{1}{3}} = \sqrt[5]{512} + \sqrt[3]{54} - \sqrt{\frac{25}{3}}$

12. $\sqrt{45c^3} - \sqrt{80c^3} + \sqrt{5a^2c} = (a - c)\sqrt{5c}$

13. $\sqrt{18a^5b^3} + \sqrt{50a^3b^3} = (3a^2b + 5ab)\sqrt{2ab}$

14. $\sqrt[3]{16a^3b} + \sqrt{4a^2b} - \sqrt{a^2b} - \sqrt[3]{54a^3b} = a\sqrt{b} - a\sqrt[3]{2b}$

15. $\sqrt[4]{2^{14}a^{13}b^5c} - \sqrt[4]{4 \cdot 5^4a^5b^9c^5} + \sqrt[4]{4 \cdot 6^4ab^5c} = (8a^3b - 5ab^2c + 6b)\sqrt[4]{4abc}$

16. $\sqrt{\frac{a^4c}{b^3}} + \sqrt{\frac{a^2c^3}{bd^2}} - \sqrt{\frac{a^2cd^2}{be^2}} = \left(\frac{a^2}{b} + \frac{ac}{d} - \frac{ad}{e}\right)\sqrt{\frac{c}{b}}$

17. $\sqrt[3]{\frac{27a^5x}{2b}} - \sqrt[3]{\frac{a^2x}{2b}} = (3a - 1)\sqrt[3]{\frac{a^2x}{2b}}$

18. $3b^2\sqrt{a^3c} + \frac{2}{c}\sqrt{a^5c^3} - c^4\sqrt{\frac{ac}{b^2}} = \left(3ab^2 + 2a^2 - \frac{c^4}{b}\right)\sqrt{ac}$

19. $5a\sqrt[3]{\frac{a^2}{b}} + b\sqrt[3]{\frac{b^2c^3}{a}} = \left(\frac{5a}{b} + \frac{bc}{a}\right)\sqrt[3]{a^2b^2}$

20. $\sqrt[3]{54a^{m+6}b^3} - \sqrt[3]{16a^{m-3}b^6} + \sqrt[3]{2a^{4m+9}} + \sqrt[3]{2c^3a^m} = (3a^2b - \frac{2b^2}{a} + a^{m+3} + c)\sqrt[3]{2a^m}$

21. $\sqrt[m]{2^ma^{mp+3}b^{mn+5}} + \sqrt[m]{3^ma^{2m-mn+3}b^{m+5}} - \sqrt[m]{a^3b^5c^{2m}} = (2a^rb^n + 3a^{2-r}b - c^2)\sqrt[m]{a^3b^5}$

22. $\sqrt[6]{\frac{3\cdot 2^3 c^3 f^4}{d^4 g}} + \sqrt[6]{\frac{2^3 g^{11}}{3^5 c^3 d^4 f^2}} = \left(\frac{f}{d} + \frac{g^2}{3cd}\right)\sqrt[6]{\frac{3\cdot 2^3 c^3 d^2}{f^2 g}}$

23. $\sqrt[2n]{\frac{a^{6n+4} b^{2n-3} c^{2mn}}{d^{3n+5} f^2 g^{2p+2n-1}}} = \frac{a^3 b c^m}{d^4 g}\sqrt[2n]{\frac{a^4}{d^{n+5} f^2 g^{2p-1} b^3}}$

24. $\sqrt{(a^2 c + a^2 d)} = a\sqrt{(c + d)}$

25. $\sqrt[3]{(a^{6m} b - a^{7m} f^2)} = a^{2m}\sqrt[3]{(b - a^m f^2)}$

26. $\sqrt[7]{\left(\frac{a^{10}}{b^6 c^6} + \frac{a^8}{b^5 c^5}\right)} = \frac{a}{bc}\sqrt[7]{(a^3 bc + ab^2 c^2)}$

27. $\sqrt{\left(\frac{a^3 b^2}{cd^2} - \frac{2a^2 b^3}{c^2 d}\right)} = \frac{ab}{cd}\sqrt{(ac - 2bd)}$

28. $x\sqrt[3]{\left(\frac{8a^4}{27b^3} + \frac{16a^3}{27b^2}\right)} = \frac{2ax}{3b}\sqrt[3]{(a + 2b)}$

29. $\sqrt{(3a^2 c + 6abc + 3b^2 c)} = (a + b)\sqrt{3c}$

30. $\sqrt{(4a^5 b^2 - 20a^3 b^3 + 25ab^4)} = (2a^2 - 5b)\sqrt{ab^2}$

31. $\sqrt{(2ax^2 - 4ax + 2a)} = (x - 1)\sqrt{2a}$

32. $\sqrt{\frac{a^3 b - 4a^2 b^2 + 4ab^3}{c^2 d^2}} = \frac{a - 2b}{cd}\sqrt{ab}$

33. $\sqrt{\frac{a^2 x - 2ax^2 + x^3}{a^2 + 2ax + x^2}} = \frac{a - x}{a + x}\sqrt{x}$

34. $\sqrt{\frac{ac}{a^2 bd - 2ab^2 d + b^3 d}} = \frac{1}{a - b}\sqrt{\frac{ac}{bd}}$

35. $\sqrt{\frac{x^3 + 2x^2 + x}{a^3 + a^2 b}} = \frac{x + 1}{a}\sqrt{\frac{x}{a + b}}$

36. $\sqrt{\frac{a^3 - a^2 x - ax^2 + x^3}{b^5 c^3 d}} = \frac{a - x}{b^2 c}\sqrt{\frac{a + x}{bcd}}$

37. $\frac{a - b}{a + b}\sqrt{\frac{ac}{a^2 - 2ab + b^2}} = \frac{\sqrt{ac}}{a + b}$

38. $\frac{a + b}{a - b}\sqrt{\frac{a - b}{a + b}} = \sqrt{\frac{a + b}{a - b}}$

39. $(x+1)\sqrt{\frac{f^2g}{x^2-1}} = \sqrt{\frac{(x+1)^2f^2g}{(x+1)(x-1)}} = \sqrt{\frac{(x+1)f^2g}{x-1}}$

40. $\sqrt[m]{\sqrt[n]{\sqrt[p]{\sqrt[q]{a}}}} = \sqrt[mnpq]{a}$

41. $\sqrt[3]{(2\sqrt{5})} = \sqrt[6]{20}$

42. $\sqrt[m]{(a\sqrt[n]{b})} = \sqrt[mn]{a^n b}$

(c) *Multiplication.*

1. $\sqrt[m]{a} \times \sqrt[m]{b} \times \sqrt[m]{c} = \sqrt[m]{abc}$

2. $a\sqrt[n]{x} \times b\sqrt[n]{y} \times c\sqrt[n]{z} = abc\sqrt[n]{xyz}$

3. $\sqrt[3]{4} \times 7\sqrt[3]{6} \times \tfrac{1}{2}\sqrt[3]{5} = \tfrac{7}{2}\sqrt[3]{120}$

4. $4 \times 2\sqrt[6]{3} \times \sqrt[6]{72} = 8\sqrt{6}$

5. $5\sqrt{3} \times 7\sqrt{\tfrac{8}{3}} \times \sqrt{2} = 140$

6. $c\sqrt{a} \times d\sqrt{a} = acd$

7. $\sqrt[m]{a} \times \sqrt[n]{b} = \sqrt[mn]{a^n} \times \sqrt[mn]{b^m} = \sqrt[mn]{a^n b^m}$

8. $\sqrt{2} \times \sqrt[3]{3} \times \sqrt[4]{5} = \sqrt[12]{648000}$

9. $\sqrt[3]{2} \times \sqrt[6]{\tfrac{1}{3}} \times \sqrt[8]{3} = \sqrt[24]{\tfrac{256}{3}}$

10. $\sqrt[5]{4} \times \sqrt[10]{3} \times \sqrt[15]{6} = \sqrt[30]{3981312}$

11. $\sqrt[7]{\tfrac{4}{3}} \times \sqrt[3]{\tfrac{1}{2}} \times \sqrt[14]{6} = \sqrt[42]{\tfrac{2}{27}}$

12. $a\sqrt[m]{x} \times b\sqrt[n]{y} \times c\sqrt[p]{z} = abc\sqrt[mnp]{x^{np}y^{mp}z^{mn}}$

13. $\sqrt[12]{\frac{a}{bc}} \times \sqrt[8]{\frac{a^m}{b}} = \sqrt[24]{\frac{a^{3m+2}}{b^5c^2}}$

14. $\frac{ac}{b^3d^3}\sqrt[3]{\frac{bcd}{e}} \times \sqrt[6]{\frac{b^{10}d^7e}{a^2c^5}} = \frac{1}{bd}\sqrt[6]{\frac{a^4c^3}{d^3e}}$

15. $(\sqrt{5}+2\sqrt{7}+3\sqrt{10})\times 2\sqrt{5}=10+4\sqrt{35}+6\sqrt{50}$

16. $(\sqrt{6}+\sqrt[3]{2}-2\sqrt[4]{5})\times\sqrt{3}=\sqrt{18}+\sqrt[6]{108}-2\sqrt[4]{45}$

17. $(3+\sqrt{5})\times(2-\sqrt{5})=1-\sqrt{5}$

18. $(7+2\sqrt{6})\times(9-5\sqrt{6})=3-17\sqrt{6}$

19. $(9-7\sqrt{13})\times(5-6\sqrt{13})=591-89\sqrt{13}$

20. $(6+12\sqrt{7})\times(3-5\sqrt{7})=6\sqrt{7}-402$

21. $(9\sqrt{12}+3)\times(5\sqrt{12}+8)=564+87\sqrt{12}$

22. $(13-\sqrt{5})\times(7+3\sqrt{5})=76+32\sqrt{5}$

23. $(\frac{3}{2}+\frac{5}{2}\sqrt{\frac{1}{2}})\times(\frac{1}{2}-7\sqrt{\frac{1}{2}})=-8-\frac{37}{4}\sqrt{\frac{1}{2}}$

24. $(-5-\sqrt{\frac{3}{4}})\times(-5+\sqrt{\frac{3}{4}})=24\frac{1}{4}$

25. $(9+2\sqrt{10})\times(9-2\sqrt{10})=41$

26. $(\sqrt{2}+\sqrt{3})\times(2\sqrt{2}-\sqrt{3})=1+\sqrt{6}$

27. $(5\sqrt{14}+3\sqrt{5})\times(7\sqrt{14}-2\sqrt{5})=460+11\sqrt{70}$

28. $(2\sqrt{7}-5\sqrt{6})\times\left(\frac{3\sqrt{7}}{2}-2\sqrt{6}\right)=81-\frac{23}{2}\sqrt{42}$

29. $(4\sqrt{\frac{7}{3}}+5\sqrt{\frac{1}{2}})\times(\sqrt{\frac{7}{3}}+2\sqrt{\frac{1}{2}})=\frac{43}{3}+13\sqrt{\frac{7}{6}}$

30. $(\sqrt{2}+\sqrt{3})^3=11\sqrt{2}+9\sqrt{3}$

31. $(\sqrt{7}-\sqrt{3})\times(\sqrt{5}-\sqrt{2})=\sqrt{35}-\sqrt{15}-\sqrt{14}+\sqrt{6}$

32. $(5-8\sqrt{7})\times(9+10\sqrt{3})=45-72\sqrt{7}+50\sqrt{3}-80\sqrt{21}$

33. $(7\sqrt{6}+2\sqrt{3})\times(\sqrt{5}+\sqrt{6})=7\sqrt{30}+2\sqrt{15}+42+2\sqrt{18}$

34. $(3\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{3}})\times(5\sqrt{\frac{5}{6}}+\sqrt{\frac{2}{7}})=15\sqrt{\frac{5}{12}}-5\sqrt{\frac{5}{18}}+3\sqrt{\frac{1}{7}}-\sqrt{\frac{2}{21}}$

35. $(5\sqrt{3}-7\sqrt{6})\times(2\sqrt{8}-3)=41\sqrt{6}-71\sqrt{3}$

36. $(2\sqrt{6}-3\sqrt{5}) \times (4\sqrt{3}-\sqrt{10}) = 39\sqrt{2}-16\sqrt{15}$

37. $(\sqrt{12}-2\sqrt{7}) \times (2+\sqrt{21}) = 2\sqrt{7}-10\sqrt{3}$

38. $(3\sqrt{5}+2\sqrt{6}-2) \times (2\sqrt{5}+18\sqrt{6}) = 246 + 58\sqrt{30} - 4\sqrt{5} - 36\sqrt{6}$

39. $(2\sqrt{8}+3\sqrt{5}-7\sqrt{2}) \times (\sqrt{72}-5\sqrt{20}-2\sqrt{2}) = -174 + 42\sqrt{10}$

40. $(2\sqrt{5}+3\sqrt{2}-8\sqrt{6}) \times (2+5\sqrt{2}-3\sqrt{12}) = 30 + 4\sqrt{5} + 150\sqrt{2} - 34\sqrt{6} + 10\sqrt{10} - 40\sqrt{12} - 6\sqrt{60}$

41. $(3\sqrt{2}+2\sqrt{5}+\sqrt{7}) \times (\sqrt{6}+5\sqrt{3}+\sqrt{10}) = 3\sqrt{12}+2\sqrt{30}+\sqrt{42}+15\sqrt{6}+10\sqrt{15} + 5\sqrt{21}+3\sqrt{20}+2\sqrt{50}+\sqrt{70}$

42. $(\sqrt[3]{5}-2\sqrt[3]{6}) \times (3\sqrt[3]{4}-\sqrt[3]{36}) = 12+3\sqrt[3]{20} - 6\sqrt[3]{24}-\sqrt[3]{180}$

43. $(5\sqrt[3]{4}-2\sqrt[3]{16}) \times (2\sqrt[3]{2}-3\sqrt[3]{4}) = 44-4\sqrt[3]{32} - 15\sqrt[3]{16}$

44. $(2\sqrt{3}+\sqrt[3]{2}) \times (2+\sqrt[3]{9}) = 4\sqrt{3}+2\sqrt[3]{2}+\sqrt[3]{18} + 6\sqrt[6]{3}$

45. $(5+\sqrt[3]{4}+2\sqrt[4]{5}) \times (\sqrt{6}+\sqrt{5}) = 5\sqrt{6}+5\sqrt{5} + 2\sqrt[4]{125}+2\sqrt[4]{180}+2\sqrt[6]{54}+\sqrt[6]{2000}$

46. $(a+\sqrt{b}) \times (a-\sqrt{b}) = a^2 - b$

47. $(\sqrt{a}+\sqrt{b}) \times (\sqrt{a}-\sqrt{b}) = a - b$

48. $(c\sqrt{a}+d\sqrt{b}) \times (c\sqrt{a}-d\sqrt{b}) = ac^2 - bd^2$

49. $(a+\sqrt{x}) \times (b+\sqrt{y}) = ab+a\sqrt{y}+b\sqrt{x}+\sqrt{xy}$

50. $\left(\sqrt{\frac{ad^2}{c^3}} + \sqrt{\frac{a^2}{b}}\right) \times (\sqrt{ac} + \sqrt{b^3}) = \frac{ad}{c} + ab + \left(a + \frac{b^2d}{c^3}\right)\sqrt{\frac{ac}{b}}$

51. $\left(\sqrt{\frac{ac^2}{a+b}} + \sqrt{\frac{1}{b}}\right) \times \left(\frac{c}{d}\sqrt{(a+b)\,a} - \sqrt{\frac{b^5}{c^2}}\right) = \frac{ac^2}{d} - \frac{b^2}{c} + \left(\frac{c}{bd} - \frac{b^2}{a+b}\right)\sqrt{(a+b)\,ab}$

52. $(\sqrt{a} + c\sqrt[3]{b}) \times (\sqrt{a} - c\sqrt[3]{b}) = a - c^2\sqrt[3]{b^2}$

53. $(2\sqrt{a} + 3c\sqrt[3]{b}) \times (\sqrt{a} + 4\sqrt[3]{b}) = 2a + 12c\sqrt[3]{b^2} + (3c + 8)\sqrt[6]{a^3b^2}$

54. $(c\sqrt[4]{a} + d\sqrt[4]{b}) \times (f\sqrt[4]{a} + g\sqrt[4]{b}) = cf\sqrt{a} + dg\sqrt{b} + (df + cg)\sqrt[4]{ab}$

55. $(\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c})^2 = \sqrt{a} + \sqrt{b} + \sqrt{c} + 2\sqrt[4]{ab} + 2\sqrt[4]{ac} + 2\sqrt[4]{bc}$

56. $\sqrt{(a + \sqrt{b})} \times \sqrt{(c + \sqrt{d})} = \sqrt{(ac + c\sqrt{b} + a\sqrt{d} + \sqrt{bd})}$

57. $\sqrt[n]{(a + \sqrt{b})} \times \sqrt[n]{(a - \sqrt{b})} = \sqrt[n]{(a^2 - b)}$

58. $\sqrt[m]{(a + \sqrt[n]{b})} \times \sqrt[m]{(c + \sqrt[p]{d})} = \sqrt[m]{(ac + c\sqrt[n]{b} + a\sqrt[p]{d} + \sqrt[np]{b^p d^n})}$

59. $\sqrt[4]{(5 + 2\sqrt{6})} \times \sqrt{(3 + \sqrt{6})} = \sqrt[4]{(147 + 60\sqrt{6})}$

60. $3\sqrt[n]{(2 + 4\sqrt[3]{3})} \times 4\sqrt[n]{6 + 2\sqrt[3]{9}} = 12\sqrt[n]{(36 + 4\sqrt[3]{9} + 24\sqrt[3]{3})}$

61. $5\sqrt{2} \times 3\sqrt{(4 + 6\sqrt{2})} = 30\sqrt{(2 + 3\sqrt{2})}$

(d) Division.

1. $\sqrt[m]{a} : \sqrt[m]{b} = \sqrt[m]{\frac{a}{b}}$

2. $c\sqrt[m]{a} : d\sqrt[m]{b} = \frac{c}{d}\sqrt[m]{\frac{a}{b}}$

3. $a : \sqrt[m]{b} = \sqrt[m]{\frac{a^m}{b}}$

4. $a : \sqrt{a} = \sqrt{a}$

5. $2ab^2c^3 : 4\sqrt[3]{a^3bc^5d} = \frac{1}{2}\sqrt[3]{\frac{b^5c^4}{d}}$

6. $\sqrt[5]{ab^{n-1}c^2} : \sqrt[5]{\frac{a^3b^2}{dc_{n-1}}} = \sqrt[5]{\frac{b^{n-3}c^{n+1}d}{a^2}}$

7. $\sqrt[n]{\frac{f^3g^2}{dx^3}} : \sqrt[n]{\frac{fg}{dx}} = \sqrt[n]{\frac{f^2g}{x^4}}$

8. $\sqrt[3]{a^2bc} : \sqrt[5]{ab^2c^3} = \sqrt[15]{\frac{a^7}{bc^4}}$

9. $\sqrt[4]{\frac{a}{b}} : \sqrt{\frac{a}{b}} = \sqrt[4]{\frac{b}{a}}$

10. $4\sqrt[3]{12} : 2\sqrt{3} = 2\sqrt[6]{\frac{16}{3}}$

11. $\sqrt[5]{64} : 2 = \sqrt[5]{2}$

12. $\sqrt[2n]{\frac{a^mb}{c^2d}} : \sqrt[3n]{\frac{a^{m-1}c^3}{d^5}} = \sqrt[6n]{\frac{a^{m+2}b^3d^7}{c^{12}}}$

13. $c\sqrt{(a^2 - x^2)} : \sqrt{(a + x)} = c\sqrt{(a - x)}$

14. $\sqrt{(ab^2 - b^2c)} : \sqrt{(a - c)} = b$

15. $\sqrt{(a^2 - z^2)} : (a - z) = \sqrt{\frac{a + z}{a - z}}$

16. $(\sqrt{72} + \sqrt{32} - 4) : \sqrt{8} = 5 - \sqrt{2}$

17. $(\sqrt{6} + 4\sqrt{18} - 3 - 8\sqrt{2}) : \sqrt{3} = \sqrt{2} + 4\sqrt{6} - \sqrt{3} - 8\sqrt{\frac{2}{3}}$

18. $(3\sqrt{15} - \sqrt{20} + \sqrt{10} - 7) : 2\sqrt{5} = \frac{3}{2}\sqrt{3} - 1 + \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{\frac{49}{5}}$

19. $(2\sqrt{32} + 3\sqrt{2} + 4) : 4\sqrt{8} = \frac{11}{8} + \frac{1}{4}\sqrt{2}$

20. $(6 + 2\sqrt{3} - \sqrt[3]{18}) : \sqrt{6} = \sqrt{6} + \sqrt{2} - \sqrt[6]{\frac{3}{2}}$

21. $(\sqrt{8} + \sqrt[3]{12} + \sqrt[4]{2}) : 2\sqrt{2} = 1 + \frac{\sqrt[6]{18}}{2} + \frac{\sqrt[4]{8}}{4}$

22. $1 : (\sqrt{3} + 2) = 2 - \sqrt{3}$

23. $3 : (1 + \sqrt{2}) = 3\sqrt{2} - 3$

24. $12 : (5 - \sqrt{21}) = 15 + 3\sqrt{21}$

25. $7 : (\sqrt{8} - 2) = \frac{7}{2}(\sqrt{2} + 1)$

26. $\sqrt{3} : (2\sqrt{5} - 3\sqrt{2}) = \sqrt{15} + \frac{3}{2}\sqrt{6}$

27. $\frac{3}{4}\sqrt{\frac{5}{6}} : (\sqrt{\frac{1}{2}} - 2) = -\frac{\sqrt{15} + 2\sqrt{30}}{28}$

28. $\frac{1}{2}\sqrt{\frac{1}{2}} : (\sqrt{2} + 3\sqrt{\frac{1}{2}}) = \frac{1}{10}$

29. $(1 + \sqrt{2}) : (2 - \sqrt{2}) = 2 + \frac{3}{2}\sqrt{2}$

30. $(5 - 7\sqrt{3}) : (1 + \sqrt{3}) = 6\sqrt{3} - 13$

31. $(6 - 3\sqrt{5}) : (\sqrt{5} - 1) = \frac{3}{4}\sqrt{5} - \frac{9}{4}$

32. $(\sqrt{3} + \sqrt{2}) : (\sqrt{3} - \sqrt{2}) = 5 + 2\sqrt{6}$

33. $(3\sqrt{5} - 2\sqrt{2}) : (2\sqrt{5} - \sqrt{18}) = 9 + \frac{5}{2}\sqrt{10}$

34. $(6\sqrt{7} - 3\sqrt{3}) : (\sqrt{5} - 2) = 6\sqrt{35} + 12\sqrt{7} - 3\sqrt{15} - 6\sqrt{3}$

35. $1 : (\sqrt{2} + \sqrt{3} - \sqrt{5}) = \frac{\sqrt{30}}{12} + \frac{\sqrt{2}}{4} + \frac{\sqrt{3}}{6}$

36. $7 : (\sqrt{10} - \sqrt{2} - \sqrt{3}) = 35\sqrt{10} + 77\sqrt{2} + 63\sqrt{3} + 14\sqrt{60}$

37. $\sqrt{2} : (1 + 2\sqrt{2} - \sqrt{5}) = \frac{3}{4}\sqrt{2} + \frac{1}{2}\sqrt{5} - \frac{1}{4}\sqrt{10} - \frac{1}{2}$

38. $(2 - \sqrt{3}) : (1 + \sqrt{2} + \sqrt{3}) = 1 + \frac{5}{4}\sqrt{2} - \frac{1}{2}\sqrt{3} - \frac{3}{4}\sqrt{6}$

39. $(3 + 4\sqrt{3}) : (\sqrt{6} + \sqrt{2} - \sqrt{5}) = \sqrt{6} + \sqrt{2} + \sqrt{5}$.

40. $(156 + 12\sqrt{11}) : (6 + 14\sqrt{2} - 2\sqrt{11}) = 7\sqrt{2} + \sqrt{11} - 3$

41. $(2\sqrt{6} + 3\sqrt{10}) : (3\sqrt{2} - \sqrt{3} + \sqrt{5}) = \frac{7}{10}\sqrt{30} + \frac{27}{10}\sqrt{5} - \frac{5}{2}\sqrt{3} - 3\sqrt{2}$

42. $\sqrt{a} : (b + \sqrt{c}) = \dfrac{b\sqrt{a} - \sqrt{ac}}{b^2 - c}$

43. $\sqrt{a} : (\sqrt{b} + \sqrt{c}) = \dfrac{\sqrt{ab} - \sqrt{ac}}{b - c}$

44. $(c\sqrt{a} + d\sqrt{b}) : (f\sqrt{h} + g\sqrt{l}) = \dfrac{cf\sqrt{ah} + df\sqrt{bh} - cg\sqrt{al} - dg\sqrt{bl}}{hf^2 - lg^2}$

45. $[(f^2 - hg^2 - m)\sqrt{m} - 2gm\sqrt{h}] : (f + g\sqrt{h} + \sqrt{m}) = f\sqrt{m} - g\sqrt{hm} - m$

46. $1 : \sqrt[n]{(a + \sqrt{b})} = \sqrt[n]{\dfrac{a - \sqrt{b}}{a^2 - b}}$

47. $\sqrt[n]{(\sqrt{a} + \sqrt{b})} : \sqrt[n]{(\sqrt{a} - \sqrt{b})} = \sqrt[n]{\dfrac{a + b + 2\sqrt{ab}}{a - b}}$

48. $1 : (\sqrt[4]{a} + \sqrt[4]{b}) = \dfrac{\sqrt[4]{a^3} - \sqrt[4]{a^2 b} + \sqrt[4]{ab^2} - \sqrt[4]{b^3}}{a - b}$*

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* The Divisor and Dividend are multiplied by $\sqrt[4]{a} - \sqrt[4]{b}$, and then treated as usual.

49. $(\sqrt[4]{a} + \sqrt[4]{b}) : (\sqrt[4]{a} - \sqrt[4]{b}) = \dfrac{a + b + 2\sqrt{ab} + 2\sqrt[4]{a^3b} + 2\sqrt[4]{ab^3}}{a - b}$

50. $\sqrt{(ab + \sqrt{af})} : \sqrt{a} = \sqrt{\left(b + \sqrt{\dfrac{f}{a}}\right)}$

(e) Square Root of a Binomial of the form $A \pm \sqrt{B}$.

FORMULA.

$$\sqrt{(A \pm \sqrt{B})} = \sqrt{\frac{A + \sqrt{(A^2 - B)}}{2}} \pm \sqrt{\frac{A - \sqrt{(A^2 - B)}}{2}}$$

1. $\sqrt{(7 + 4\sqrt{3})} = 2 + \sqrt{3}$
2. $\sqrt{(43 - 15\sqrt{8})} = 5 - 3\sqrt{2}$
3. $\sqrt{(5 - \sqrt{24})} = \sqrt{3} - \sqrt{2}$
4. $\sqrt{(3 - 2\sqrt{2})} = \sqrt{2} - 1$
5. $\sqrt{(28 + 5\sqrt{12})} = 5 + \sqrt{3}$
6. $\sqrt{(87 - 12\sqrt{42})} = 3\sqrt{7} - 2\sqrt{6}$
7. $\sqrt{(\frac{3}{2} + \sqrt{2})} = 1 + \frac{1}{2}\sqrt{2}$
8. $\sqrt{(2 + \sqrt{3})} = \frac{1}{2}\sqrt{6} + \frac{1}{2}\sqrt{2}$
9. $\sqrt{(\sqrt{27} + 2\sqrt{6})} = \sqrt[4]{12} + \sqrt[4]{3}$*
10. $\sqrt{(\sqrt{32} - \sqrt{24})} = \sqrt[4]{18} - \sqrt[4]{2}$
11. $\sqrt{(3\sqrt{5} + \sqrt{40})} = \sqrt[4]{20} + \sqrt[4]{5}$

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* $A = \sqrt{27}$.

Ex. 9.

$\sqrt{(\sqrt{27} + 2\sqrt{6})} = \sqrt{(3\sqrt{3} + 2\sqrt{3}\cdot\sqrt{2})} = \sqrt[4]{3}\sqrt{(3 + 2\sqrt{2})}$

$= \sqrt[4]{3}(\sqrt{2} + 1) = \sqrt[4]{12} + \sqrt[4]{3}$

12. $\sqrt{(3\sqrt{6} + 2\sqrt{12})} = \sqrt[4]{24} + \sqrt[4]{6}$

13. $\sqrt{(\sqrt{18} - 4)} = \sqrt[4]{8} - \sqrt[4]{2}$

14. $\sqrt{(a^2 + b + 2\sqrt{b})} = a + \sqrt{b}$

15. $\sqrt{(ac^2 + bd^2 + 2cd\sqrt{ab})} = c\sqrt{a} + d\sqrt{b}$

16. $\sqrt{[2a + 2\sqrt{(a^2 - b^2)}]} = \sqrt{(a+b)} + \sqrt{(a-b)}$

17. $\sqrt{[x - 2\sqrt{(x-1)}]} = \sqrt{(x-1)} - 1$

18. $\sqrt{\left[\frac{a^2}{4} + \frac{c}{2}\sqrt{(a^2 - c^2)}\right]} = \frac{c + \sqrt{(a^2 - c^2)}}{2}$

19. $\sqrt{(x + xy - 2x\sqrt{y})} = (\sqrt{y} - 1)\sqrt{x}$

20. $\sqrt{[ap - 2a\sqrt{(ap - a^2)}]} = \sqrt{(ap - a^2)} - a$

21. $\sqrt{\left[\frac{3a}{b} + \sqrt{\left(\frac{12a^3c^2}{bd^2} - \frac{4a^4c^4}{d^4}\right)}\right]} = \frac{ac}{d} + \sqrt{\left(\frac{3a}{b} - \frac{a^2c^2}{d^2}\right)}$

22. $\sqrt{[b^2 - ab + \frac{a^2}{4} + \sqrt{(4ab^3 - 8a^2b^2 + a^3b)}]} =$

$$\sqrt{ab} + \sqrt{\left(b^2 - 2ab + \frac{a^2}{4}\right)}$$

V. *The Expression of Roots by Fractional Powers, and their Arithmetic.*

A Power with a Fractional Exponent may be considered as an interpolated term of a series of Powers having Integer Exponents; yet the usual way in which a Fractional Exponent denotes the raising of a Root to a Power, seems to me indeed to be the easiest of comprehension for beginners. The whole of this Theory, as well as that of logarithms, founded upon it, may be exposed

Ex. 13. $\sqrt{\sqrt{18} - 4} = \sqrt{3\sqrt{2} - \sqrt{8}\cdot\sqrt{2}} = \sqrt{2}\cdot\sqrt{3 - \sqrt{8}}$ &c. &c.

with the accuracy of Euclid, and merely by signs. I submit, however, these considerations to the examination of he Scientific, not considering my own judgment as decisive.

(1.) EXPRESSIONS.

1. $\sqrt[m]{a^n} = a^{\frac{n}{m}}$

2. $\dfrac{1}{\sqrt[m]{a^n}} = a^{-\frac{n}{m}}.$

3. $\sqrt[m]{a^n b^p c^q} = a^{\frac{n}{m}} b^{\frac{p}{m}} c^{\frac{q}{m}} = (a^n b^p c^q)^{\frac{1}{m}}$

4. $\sqrt[m]{\dfrac{a^n b^p}{c^r d^s e^t}} = a^{\frac{n}{m}} b^{\frac{p}{m}} c^{-\frac{r}{m}} d^{-\frac{s}{m}} e^{-\frac{t}{m}}$

5. $c\sqrt[4]{a^3} + \dfrac{d}{\sqrt[7]{a^2}} = ca^{\frac{3}{4}} + da^{-\frac{2}{7}}$

6. $\sqrt[5]{a^2bc} = a^{\frac{2}{5}} b^{\frac{1}{5}} c^{\frac{1}{5}} = (a^2bc)^{\frac{1}{5}}$

7. $\sqrt[6]{\dfrac{a^5b^7}{c^{12}}} + \sqrt[8]{\dfrac{a^6b^4}{d^{20}}} = a^{\frac{5}{6}} b^{\frac{7}{6}} c^{-2} + a^{\frac{3}{4}} b^{\frac{1}{2}} d^{-\frac{3}{2}}$

8. $\dfrac{\sqrt{(c+d)}}{\sqrt[3]{c^5}} = (c+d)^{\frac{1}{2}} c^{-\frac{5}{3}}$

9. $\dfrac{\sqrt{(a^2-x^2)}}{\sqrt{a}\cdot\sqrt[3]{(a+x)}} = a^{-\frac{1}{2}}(a+x)^{-\frac{1}{3}}(a^2-x^2)^{\frac{1}{2}}$

10. $\dfrac{\sqrt[3]{(a+b)^7c^4}}{\sqrt[5]{f^2}\cdot\sqrt[6]{g^2}} = \dfrac{c^{\frac{4}{3}}(a+b)^{\frac{7}{3}}}{f^{\frac{2}{5}}g^{\frac{1}{3}}} = c^{\frac{4}{3}} f^{-\frac{2}{5}} g^{-\frac{1}{3}} (a+b)^{\frac{7}{3}}$

(2.) ARITHMETIC OF FRACTIONAL POWERS.

(*a*) *Multiplication.**

1. $a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}} = a^{\frac{mq+np}{nq}}$

2. $a^{\frac{m}{n}} \times a^{-\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}} = a^{\frac{mq-np}{nq}}$

3. $a^{-\frac{m}{n}} \times a^{-\frac{p}{q}} = a^{-\frac{m}{n}-\frac{p}{q}} = a^{-\left(\frac{m}{n}+\frac{p}{q}\right)} = a^{-\frac{mq+np}{nq}}$

4. $a^{\frac{3}{4}} \times a^{\frac{5}{6}} = a^{\frac{9}{12}+\frac{10}{12}} = a^{\frac{19}{12}}\sqrt{a^5}$

5. $a^{-\frac{1}{2}} \times a^{\frac{7}{4}} \times a^{-\frac{1}{5}} = a^{\frac{20}{40}-\frac{10}{40}+\frac{31}{40}} = a\sqrt[40]{a}$

6. $a^{-\frac{3}{4}} \times a^{-\frac{7}{8}} = a^{-\frac{13}{8}} = \dfrac{1}{a\sqrt[8]{a^5}}$

7. $a^{-\frac{3}{4}}b^{-2} \times a^{\frac{5}{6}}b^{\frac{1}{2}}c = a^{\frac{1}{12}}b^{-\frac{3}{2}}c = \dfrac{c}{b}\sqrt[12]{\dfrac{a}{b^6}}$

8. $\dfrac{a}{b^{\frac{1}{2}}c^{\frac{3}{4}}} \times \dfrac{a^{\frac{7}{8}}b}{c^{-\frac{1}{2}}} = a^{\frac{15}{8}}b^{\frac{1}{2}}c^{-\frac{1}{4}} = a^2\sqrt[8]{\dfrac{b^4}{ac^2}}$

9. $\sqrt[5]{a^{12}} \times \sqrt[7]{a^3} \times \sqrt[6]{a^4} = a^{\frac{12}{5}} \cdot a^{\frac{3}{7}} \cdot a^{\frac{2}{3}} = a^{\frac{36}{105}+\frac{15}{105}+\frac{70}{105}} = a^3\sqrt[105]{a^{52}}$

10. $\sqrt[5]{\sqrt[3]{a^2}} \times \sqrt[6]{\sqrt[4]{a^9}} = a^{\frac{2}{15}} \cdot a^{\frac{3}{8}} = \sqrt[120]{a^{61}} = \sqrt{a} \cdot \sqrt[120]{a}$

11. $\sqrt[5]{\dfrac{(c^2-y^2)^3}{(a+x)^3}} \times \sqrt[6]{\dfrac{(c^2-y^2)^{\frac{3}{2}}}{a+x}} = (c^2-y^2)^{\frac{17}{20}}(a+x)^{-\frac{53}{30}}$
$= \dfrac{c^2-y^2}{(a+x)^2}\sqrt[60]{\dfrac{(a+x)^{14}}{(c^2-y^2)^9}}$

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* The Addition and Subtraction of Fractional Powers, involving no peculiar difficulties, are here omitted.

12. $\frac{b}{\sqrt{a}} \times \sqrt[3]{ac} \times \frac{\sqrt[4]{c^3}}{\sqrt{b}} = ba^{-\frac{1}{2}} \times a^{\frac{1}{3}}c^{\frac{1}{3}} \times c^{\frac{3}{4}}b^{-\frac{1}{2}} =$
$a^{-\frac{1}{6}}a^{\frac{1}{2}}c^{\frac{1}{1\cdot 2}} = c\sqrt[12]{\frac{b^6c}{a^2}}$

13. $(\sqrt[4]{a^3} + \sqrt[5]{b^2}) \times (\sqrt[4]{a^3} - \sqrt[5]{b^2}) = (a^{\frac{3}{4}} + b^{\frac{2}{5}}) \times$
$(a^{\frac{3}{4}} - b^{\frac{2}{5}}) = a^{\frac{3}{2}} - b^{\frac{4}{5}} = a\sqrt{a} - \sqrt[5]{b^4}$

14. $\left(5\sqrt[4]{a^7} - \frac{6ab}{\sqrt[4]{a}}\right) \times \left(\sqrt[3]{a} - \frac{7b}{\sqrt[3]{a^2}}\right) = (5a^{\frac{7}{4}} - 6a^{\frac{3}{4}}b)$
$\times (a^{\frac{1}{4}} - 7a^{-\frac{2}{4}}b) = 5a^{\frac{2}{1}\frac{5}{2}} - 41a^{\frac{1}{1}\frac{3}{2}}b + 42a^{\frac{1}{1\cdot 2}}b^2$
$= (5a^2 - 41ab + 42b^2)\sqrt[12]{a}$

15. $\left(\sqrt[5]{ab^3} + 3\sqrt[5]{\frac{b^8}{a^4}}\right) \times \left(\sqrt{ab} + \frac{2}{b^2}\sqrt{\frac{b}{a}}\right) = (a^{\frac{1}{2}}b^{\frac{3}{5}} +$
$3b^{\frac{6}{5}}a^{-\frac{4}{5}}) \times (a^{\frac{1}{2}}b^{\frac{1}{2}} + 2a^{-\frac{1}{2}}b^{-\frac{3}{2}}) = a^{\frac{7}{1\cdot 0}}b^{\frac{1}{1\cdot 0}} +$
$2a^{-\frac{3}{1\cdot 0}}b^{-\frac{9}{1\cdot 0}} + 3a^{-\frac{3}{1\cdot 0}}b^{\frac{2}{1\cdot 0}} + 6a^{-\frac{1}{1\cdot 0}}b^{\frac{1}{1\cdot 0}} = (ab +$
$\frac{2}{b} + 3b^2 + \frac{6}{a}\Big)\sqrt[10]{\frac{b}{a^3}}$

16. $\left(\sqrt[6]{\frac{1}{a^3b^2}} - \frac{2\sqrt[3]{b^2c^3}}{a\sqrt{a}}\right) \times \left(\sqrt[5]{a^2} - \frac{b}{\sqrt[5]{a^3}}\right) = (a^{-\frac{1}{2}}b^{-\frac{1}{3}}$
$-2a^{-\frac{3}{2}}b^{\frac{2}{3}}c) \times (a^{\frac{2}{5}} - a^{-\frac{3}{5}}b) = a^{-\frac{1}{1\cdot 0}}b^{-\frac{1}{3}} - 2a^{-\frac{1}{1\cdot 0}}b^{\frac{2}{3}}c$
$-a^{-\frac{1}{1\cdot 0}}b^{\frac{2}{3}} + 2a^{-\frac{2}{1\cdot 0}}b^{\frac{5}{3}}c = \left(1 - \frac{2bc}{a} - \frac{b}{a} + \frac{2b^2c}{a^2}\right) \times$
$\sqrt[30]{\frac{1}{a^3b^{10}}}$

17. $\left(\frac{b}{c}\sqrt[5]{\frac{ad}{f}} - cd\sqrt[3]{\frac{ac}{bg}}\right) \times \left(\sqrt[5]{\frac{ab^5d}{c^5f}} + \sqrt[3]{\frac{ac^4d^3}{bg}}\right) =$

$$\left[\frac{(ad)^{\frac{1}{5}}b}{cf^{\frac{1}{5}}}-\frac{(ac)^{\frac{1}{3}}cd}{(bg)^{\frac{1}{3}}}\right]\times\left[\frac{(ad)^{\frac{1}{5}}b}{cf^{\frac{1}{5}}}+\frac{(ac)^{\frac{1}{3}}cd}{(bg)^{\frac{1}{3}}}\right]=$$

$$\frac{(ad)^{\frac{2}{5}}b^2}{c^2f^{\frac{2}{5}}}-\frac{(ac)^{\frac{2}{3}}c^2d^2}{(bg)^{\frac{2}{3}}}=\frac{b^2}{c^2}\sqrt[5]{\frac{a^2d^2}{f^2}}-c^2d^2\sqrt[3]{\frac{a^2c^2}{b^2g^2}}$$

(*b*) *Division.*

1. $a^{\frac{m}{n}}:a^{\frac{p}{q}}=a^{\frac{m}{n}-\frac{p}{q}}=a^{\frac{mq-np}{nq}}$

2. $a^{\frac{m}{n}}:a^{-\frac{p}{q}}=a^{\frac{m}{n}+\frac{p}{q}}=a^{\frac{mq+np}{nq}}$

3. $a^{-\frac{m}{n}}:a^{\frac{p}{q}}=a^{-\frac{m}{n}-\frac{p}{q}}=a^{-\left(\frac{m}{n}+\frac{p}{q}\right)}=a^{-\frac{mq+np}{nq}}$

4. $a^{-\frac{m}{n}}:a^{-\frac{p}{q}}=a^{\frac{p}{q}-\frac{m}{n}}=a^{\frac{np-mq}{nq}}=a^{-\frac{mq-np}{nq}}$

5. $ca^{\frac{3}{4}}:da^{\frac{5}{6}}=\dfrac{ca^{-\frac{1}{12}}}{d}=\dfrac{c}{d\sqrt[12]{a}}$

6. $a^{\frac{2}{5}}b^{\frac{1}{2}}:a^{-\frac{7}{5}}b^{-\frac{1}{4}}c=\dfrac{a^2b^{\frac{3}{4}}}{c}=\dfrac{a^2}{c}\sqrt[4]{b^3}$

7. $h:\dfrac{a^{\frac{1}{3}}b^{\frac{1}{3}}}{cd^{\frac{1}{2}}}=\dfrac{ch\sqrt{d}}{\sqrt[12]{a^4b^3}}$

8. $\dfrac{a^{-\frac{2}{3}}b^{\frac{2}{3}}}{c^{\frac{1}{6}}d^3}:\dfrac{a^{-\frac{2}{3}\cdot\frac{9}{4}}d^{\frac{1}{3}\cdot\frac{1}{3}}}{b^{\frac{2}{3}}c}=\dfrac{a^{\frac{1}{4}\cdot\frac{1}{3}}b^{\frac{3}{4}\cdot\frac{4}{15}}c}{c^{\frac{1}{6}}d^{\frac{2}{3}\cdot\frac{10}{3}}}=\dfrac{a^3b^2c}{d^7}\sqrt[60]{\dfrac{b^{16}d^{20}}{a^{15}c^{10}}}=$

$$\frac{a^3b^2c}{d^7}\sqrt[12]{\frac{d^4}{a^3c^2}}\cdot\sqrt[15]{b^4}$$

9. $(a^3-2\sqrt[4]{a^2b^3}-a^2\sqrt[6]{a^3b^2}+2b\sqrt[12]{b}):(\sqrt{a}-\sqrt[3]{b})$

$=(a^3-2a^{\frac{1}{2}}b^{\frac{3}{4}}-a^{\frac{5}{3}}b^{\frac{1}{3}}+2b^{\frac{13}{12}}):(a^{\frac{1}{2}}-b^{\frac{1}{3}})=$

$a^{\frac{3}{2}}-2b^{\frac{3}{4}}=a^2\sqrt{a}-2\sqrt[4]{b^3}$

10. $\left(\sqrt[12]{a^{10}b^9} - c\sqrt[10]{a^7}\cdot\sqrt[6]{b^5} - \frac{5}{2}a\sqrt[4]{b^3} + \frac{3abc}{2}\sqrt[30]{\frac{1}{a^4b^5}}\right)$
$: (\sqrt{ab} - \frac{5}{2}\sqrt[6]{a^4b^3}) = (a^{\frac{5}{6}}b^{\frac{3}{4}} - ca^{\frac{7}{10}}b^{\frac{5}{6}} - \frac{5}{2}ab^{\frac{3}{4}}$
$+ \frac{5}{2}ca^{\frac{3}{10}}b^{\frac{5}{6}}) : (a^{\frac{1}{2}}b^{\frac{1}{2}} - \frac{5}{2}a^{\frac{2}{3}}b^{\frac{1}{2}}) = a^{\frac{1}{3}}b^{\frac{1}{4}} - ca^{\frac{1}{5}}b^{\frac{1}{3}}$
$= \sqrt[12]{a^4b^3} - c\sqrt[15]{a^3b^5}$

11. $(5a^2 - 41ab + 42b^2)\sqrt[12]{a} : \left(\sqrt[3]{a} - \frac{7b}{\sqrt[3]{a^2}}\right) = (5a^{\frac{2}{12}}$
$- 41a^{\frac{1}{12}}b + 42a^{\frac{1}{12}}b^2) : (a^{\frac{1}{3}} - 7ba^{-\frac{2}{3}}) = 5a^{\frac{7}{4}}$
$- 6a^{\frac{3}{4}}b = (5a - 6b)\sqrt[4]{a^3}$

12. $(\sqrt[4]{a^3} - \sqrt[4]{b^3}) : (\sqrt[4]{a} - \sqrt[4]{b}) = (a^{\frac{3}{4}} - b^{\frac{3}{4}}) :$
$(a^{\frac{1}{4}} - b^{\frac{1}{4}}) = a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}} = \sqrt{a} + \sqrt{b} + \sqrt[4]{ab}$

(c) *Powers of Powers.*

1. $\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}} = \sqrt[q]{(\sqrt[n]{a^m})^p} = a^{\frac{mp}{nq}} = \sqrt[nq]{a^{mp}}$

2. $\left(a^{-\frac{m}{n}}\right)^{\frac{p}{q}} = \sqrt[q]{\left(\frac{1}{\sqrt[n]{a^m}}\right)^p} = a^{-\frac{mp}{nq}} = \frac{1}{\sqrt[nq]{a^{mp}}}$

3. $\left(a^{\frac{m}{n}}\right)^{-\frac{p}{q}} = \frac{1}{\sqrt[q]{(\sqrt[n]{a^m})^p}} = a^{-\frac{mp}{nq}} = \frac{1}{\sqrt[nq]{a^{mp}}}$

4. $\left(a^{-\frac{m}{n}}\right)^{-\frac{p}{q}} = \frac{1}{\sqrt[q]{\left(\frac{1}{\sqrt[n]{a^m}}\right)^p}} = a^{\frac{mp}{nq}} = \sqrt[nq]{a^{mp}}$

5. $(a^{\frac{3}{4}}b^{\frac{3}{3}})^{\frac{1}{3}} = a^{\frac{1}{4}}b^{\frac{2}{9}} = \sqrt[36]{a^9b^8}$

6. $(a^2b^{-\frac{1}{2}}c^{\frac{2}{5}})^{-\frac{1}{4}} = a^{-\frac{1}{2}}b^{\frac{1}{8}}c^{-\frac{1}{10}} = \sqrt[40]{\frac{b^5}{a^{20}c^4}}$

7. $[(a^{\frac{1}{2}})^{\frac{3}{5}}]^{-\frac{1}{6}} = a^{-\frac{1}{2}\frac{1}{5}\frac{1}{6}} = \sqrt[20]{\frac{1}{a}}$

8. $\sqrt[6]{(a^3b\sqrt[5]{a^3bc})^5} = (a^{\frac{1}{2}\frac{3}{5}}b^{\frac{6}{5}}c^{\frac{1}{5}})^{\frac{5}{6}} = a^3bc^{\frac{1}{6}} = a^3b\sqrt[6]{c}$

9. $\left[\frac{c^2d}{(a+b)^{\frac{3}{2}}}\right]^{-\frac{1}{3}} = \frac{c^{-\frac{2}{3}}d^{-\frac{1}{3}}}{(a+b)^{-\frac{1}{2}}} = \frac{\sqrt{(a+b)}}{\sqrt[3]{c^2d}} = \sqrt[6]{\frac{(a+b)^3}{c^4d^2}}$

10. $\sqrt[4]{\left(\frac{a\sqrt{b}}{\sqrt[3]{ab}}\right)^3} = (a^{\frac{2}{3}}b^{\frac{1}{6}})^{\frac{3}{4}} = a^{\frac{1}{2}}b^{\frac{1}{8}} = \sqrt[8]{a^4b}$

11. $\sqrt[4]{\frac{\sqrt{(c-d)}\cdot\sqrt[3]{(a+x^2)^4}}{c^6d^{5m}}} = \left[\frac{(c-d)^{\frac{1}{2}}(a+x^2)^{\frac{4}{3}}}{c^6d^{5m}}\right]^{\frac{1}{4}} =$

$$\frac{(c-d)^{\frac{1}{8}}(a+x^2)^{\frac{1}{3}}}{c^{\frac{3}{2}}d^{\frac{5m}{4}}} = \frac{1}{cd^m}\sqrt[24]{\frac{(c-d)^3(a+x^2)^8}{c^{12}d^{6m}}}$$

VI. THE ARITHMETIC OF IMAGINARY MAGNITUDES.

It is impossible to extract an *even* root of a Negative Magnitude. Such Roots are Imaginary. In actual calculation such forms occur very often, either when it is impossible to fulfil the conditions of the question, or when the assumed form of the result is impossible. In the latter case they are mere expressions; but do not, on that account, conduct to erroneous conclusions, provided the calculation with them be effected on true algebraical principles. They are of great use in the discovery of new truths, inasmuch as they lead us to them, in spite of our-

selves, as it were, by a much nearer route than other instruments of calculation.

Now $\sqrt{-a} = \sqrt{a} \cdot \sqrt{-1}$. Moreover it may be strictly shown, that all Imaginary Magnitudes whatever may be reduced to the form $h + k\sqrt{-1}$, where h and k are real; and, under this form, calculation with them becomes very easy.

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(1) ADDITION AND SUBTRACTION.

1. $a + b\sqrt{-1} + c\sqrt{-1} - d\sqrt{-1} = a + (b + c - d)\sqrt{-1}$
2. $3\sqrt{-4} - \sqrt{-25} + 4\sqrt{-9} = 13\sqrt{-1}$
3. $2\sqrt{-48} + 3\sqrt{-12} + 5\sqrt{-8} - 7\sqrt{-32} = (14\sqrt{3} - 18\sqrt{2})\sqrt{-1}$

---

(2) MULTIPLICATION.

1. $a \times \sqrt{-a} = a\sqrt{a} \cdot \sqrt{-1}$
2. $c\sqrt{-a} \times d\sqrt{-b} = c\sqrt{a} \cdot \sqrt{-1} \times d\sqrt{b} \cdot \sqrt{-1} = -cd\sqrt{ab}$
3. $(c\sqrt{-a} + d\sqrt{-b} + f) \times \sqrt{-a} = -ac - d\sqrt{ab} + f\sqrt{a} \cdot \sqrt{-1}$
4. $(2 - \sqrt{-3}) \times (10 - \sqrt{-8}) = 20 - \sqrt{24} - (10\sqrt{3} + 4\sqrt{2})\sqrt{-1}$
5. $(7 - \sqrt{-5}) \times (10 - 3\sqrt{-6}) = 70 - 3\sqrt{30} - (10\sqrt{5} + 21\sqrt{6})\sqrt{-1}$
6. $(3 - \sqrt{-5}) \times (4 - 2\sqrt{-5}) = 2 - 10\sqrt{5} \cdot \sqrt{-1}$
7. $(2 - 5\sqrt{-3}) \times (7 - 4\sqrt{-3}) = -46 - 43\sqrt{3} \cdot \sqrt{-1}$

8. $(9+6\sqrt{-1})\times(3+7\sqrt{-1}) = -15+81\sqrt{-1}$

9. $(7-\sqrt{-\frac{1}{2}})\times(1-\sqrt{-\frac{1}{2}}) = \frac{15}{2}-4\sqrt{2}\cdot\sqrt{-1}$

10. $(1-\sqrt{-1})^2 = -2\sqrt{-1}$

11. $(\sqrt{2}-3\sqrt{-5})\times(\sqrt{7}-\sqrt{-3}) = \sqrt{14}-3\sqrt{15}-(3\sqrt{35}+\sqrt{6})\sqrt{-1}$

12. $(2\sqrt{3}-\sqrt{-5})\times(4\sqrt{3}-2\sqrt{-5}) = 14-8\sqrt{-15}$

13. $(2\sqrt{-3}-5\sqrt{-4}-7\sqrt{-2})\times(\sqrt{-7}-2\sqrt{-1}) = -2\sqrt{21}+5\sqrt{28}+7\sqrt{14}+4\sqrt{3}-20-14\sqrt{2}$

14. $(\sqrt{a}\cdot\sqrt{-1}+\sqrt{b}\cdot\sqrt{-1})^2 = -(a+b+2\sqrt{ab})$

15. $(a+\sqrt{b}\cdot\sqrt{-1})\times(a-\sqrt{b}\cdot\sqrt{-1}) = a^2+b$

16. $(a\pm\sqrt{b}\cdot\sqrt{-1})^2 = a^2-b\pm 2a\sqrt{b}\cdot\sqrt{-1}$

17. $(a\pm\sqrt{b}\cdot\sqrt{-1})^3 = a^3-3ab\pm(3a^2\sqrt{b}-b\sqrt{b})\sqrt{-1}$

18. $(a\sqrt{-1})^{4n} = a^{4n}$

19. $(a\sqrt{-1})^{4n+1} = a^{4n+1}\sqrt{-1}$

20. $(a\sqrt{-1})^{4n+2} = -a^{4n+2}$

21. $(a\sqrt{-1})^{4n+3} = -a^{4n+3}\sqrt{-1}$

(3.) DIVISION.

1. $b\sqrt{-1} : c\sqrt{-1} = \frac{b}{c}$

2. $1 : \sqrt{-1} = -\sqrt{-1}$

3. $a : b\sqrt{-1} = -\frac{a}{b}\sqrt{-1}$

4. $a : \sqrt{a}\cdot\sqrt{-1} = -\sqrt{a} : \sqrt{-1}$

5. $(\sqrt{-12}+\sqrt{-6}+\sqrt{-9}) : \sqrt{-3} = 2+\sqrt{2}+\sqrt{3}$

6. $(2\sqrt{8}-\sqrt{-10}) : -\sqrt{-2} = \sqrt{5}+4\sqrt{-1}$

7. $(3\sqrt{-4}-2\sqrt{-12}+\sqrt{6}-9) : -3\sqrt{-2} = -\sqrt{2} + \frac{2}{3}\sqrt{6}+\left(\frac{1}{3}\sqrt{3}-\frac{3}{\sqrt{2}}\right)\sqrt{-1}$

8. $6 : (1+\sqrt{-2}) = 2-2\sqrt{2}\cdot\sqrt{-1}$

9. $8 : (-1+\sqrt{-3}) = -2-2\sqrt{3}\cdot\sqrt{-1}$

10. $1 : (3-2\sqrt{-3}) = \dfrac{3+2\sqrt{3}\cdot\sqrt{-1}}{21}$

11. $14 : (4\sqrt{-3}-2\sqrt{-5}) = -(2\sqrt{3}+\sqrt{5})\sqrt{-1}$

12. $(5-\sqrt{-2}) : (1+\sqrt{-2}) = 1-2\sqrt{2}\cdot\sqrt{-1}$

13. $(4\sqrt{5}-20) : (\frac{5}{2}\sqrt{-10}-5\sqrt{-\frac{1}{2}}) = (\sqrt{10}+\sqrt{2})\times 2\sqrt{-1}$

14. $[14-\sqrt{15}-(7\sqrt{3}+2\sqrt{5})\sqrt{-1}] : (7-\sqrt{5}\cdot\sqrt{-1} = 2-\sqrt{3}\cdot\sqrt{-1}$

15. $1 : [2+(\sqrt{3}-\sqrt{5})\sqrt{-1}] = \dfrac{12+2\sqrt{15}+(3\sqrt{5}-\sqrt{3})\sqrt{-1}}{42}$

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(4) SQUARE ROOT OF A BINOMIAL OF THE FORM $A + B\sqrt{-1}$.

Formula.

$$\sqrt{(A+B\sqrt{-1})} = \sqrt{\frac{\sqrt{(A^2+B^2)}+A}{2}} + \sqrt{\frac{\sqrt{(A^2+B^2)}-A}{2}}\cdot\sqrt{-1}.$$

Examples.

1. $\sqrt{(7 + 6\sqrt{-2})} = \sqrt{(7 + 6\sqrt{2}\cdot\sqrt{-1})} = 3 + \sqrt{2}\cdot\sqrt{-1}$
2. $\sqrt{(31 + 42\sqrt{-2})} = 7 + 3\sqrt{2}\cdot\sqrt{-1}$
3. $\sqrt{(16 - 24\sqrt{-5})} = 6 - 2\sqrt{5}\cdot\sqrt{-1}$
4. $\sqrt{(-3 + \sqrt{-16})} = 1 + 2\sqrt{-1}$
5. $\sqrt{(4\sqrt{-6} - 2)} = 2 + \sqrt{6}\cdot\sqrt{-1}$
6. $\sqrt{(-83 - 60\sqrt{-3})} = 5 - 6\sqrt{3}\cdot\sqrt{-1}$
7. $\sqrt{(2 + 4\sqrt{-42})} = \sqrt{14} + 2\sqrt{3}\cdot\sqrt{-1}$
8. $\sqrt{(-2 - 2\sqrt{-15})} = \sqrt{3} - \sqrt{5}\cdot\sqrt{-1}$
9. $\sqrt{\left(\frac{a^2c}{b^2} - cd + \frac{ac\sqrt{4d}}{b}\sqrt{-1}\right)} = \frac{a}{b}\sqrt{c} + \sqrt{cd}\cdot\sqrt{-1}$
10. $\sqrt{\left(\frac{25a^2d}{c^2} - \frac{4a^2b}{d} - \frac{20a^2\sqrt{b}}{c}\sqrt{-1}\right)} = \frac{5a\sqrt{d}}{c} - 2a\sqrt{\frac{b}{d}}\cdot\sqrt{-1}$
11. $\sqrt{[a^4f^4 - a^3b^2 - a^2b^3 - 2a^3bf^2\sqrt{(a+b)}\cdot\sqrt{-1}]} = a^2f^2 - ab\sqrt{(a+b)}\cdot\sqrt{-1}$
12. $\sqrt[4]{-1} = \sqrt{(0 + \sqrt{-1})} = \sqrt{\tfrac{1}{2}} + \sqrt{\tfrac{1}{2}}\cdot\sqrt{-1}$
13. $\sqrt{(-\sqrt{-1})} = \sqrt{(0 - \sqrt{-1})} = \sqrt{\tfrac{1}{2}} - \sqrt{\tfrac{1}{2}}\cdot\sqrt{-1}$
14. $\sqrt{(8\sqrt{-1})} = \sqrt{(0 + 8\sqrt{-1})} = 2 + 2\sqrt{-1}$
15. $\sqrt{\left(\frac{2c^2}{d^2}\cdot\sqrt{-1}\right)} = \frac{c}{d}(1 + \sqrt{-1})$
16. $\sqrt{(2cd\sqrt{-1})} = (1 + \sqrt{-1})\sqrt{cd}$
17. $\sqrt{(2 + \sqrt{-3})} = \sqrt{\frac{\sqrt{7}+2}{2}} + \sqrt{\frac{\sqrt{7}-2}{2}}\cdot\sqrt{-1}$
18. $\sqrt{(5 - \sqrt{-1})} = \sqrt{\frac{\sqrt{26}+5}{2}} - \sqrt{\frac{\sqrt{26}-5}{2}}\cdot\sqrt{-1}$

VII. REDUCTIONS.

Reduction of Fractions.

1. $\frac{a}{b} + c = \frac{a + bc}{b}$

2. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

3. $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{adf + bcf + bde}{bdf}$

4. $\frac{3a}{5b} + \frac{c}{4d} + h = \frac{12ad + 5bc + 20bdh}{20bd}$

5. $\frac{a}{b} + \frac{c}{d} - \frac{e}{f} - \frac{g}{h} - k = \frac{adfh + bcfh - bdeh - bdfg - bdfhk}{bdfh}$

6. $\frac{1}{a} - \frac{1}{b} + \frac{1}{c} = \frac{bc - ac + ab}{abc}$

7. $\frac{3a}{4b} + \frac{5f}{8l} - \frac{x}{7y} = \frac{42aly + 35bfy - 8blx}{56bly}$

8. $\frac{af}{4bg} - \frac{5cd}{12bh} + \frac{2}{3} = \frac{3afh - 5cdg + 8bgh}{12bgh}$

9. $\frac{a}{4bcd} - \frac{h}{2bcg} + \frac{2cd}{5bg} = \frac{5ag - 10dh + 8c^2d^2}{20bcdg}$

10. $\frac{2a}{3bc} + \frac{5df}{8b^2c} - \frac{deg}{6b^2c^2} = \frac{16abc + 15cdf - 4deg}{24b^2c^2}$

11. $a - b - \frac{d}{ef} - \frac{c}{eg} = \frac{(a - b)efg - dg - cf}{efg}$

12. $e - f - \frac{g^3}{2ef} + \frac{f^m}{3eg} = \frac{6efg(e - f) - 3g^4 + 2f^{m+1}}{6efg}$

13. $\frac{a^2d}{3b^7c^3} - \frac{3ad}{2b^4c^2} - \frac{b^2}{cd} = \frac{2a^2d^2 - 9ab^3cd^2 - 6b^9c^2}{6b^7c^3d}$

14. $\frac{a}{b^n} + \frac{c}{b^{n-r}} + \frac{d}{b^{n-2r}} = \frac{a + cb^r + db^{2r}}{b^n}$

15. $\frac{a}{x^n} - \frac{c}{x^{n-1}} + \frac{d}{x^{n-r-s}} = \frac{a - cx + dx^{r+s}}{x^n}$

16. $c + 2ab - 3ac - \frac{b^2c - 5ab^2c + a^3}{b^2 - bc} =$

$$\frac{2ab^3 - bc^2 + 3abc^2 - a^3}{b^2 - bc}$$

17. $\frac{a+b}{2} + \frac{a-b}{2} = a$

18. $\frac{a+b}{2} - \frac{a-b}{2} = b$

19. $\frac{13a - 5b}{4} - \frac{7a - 2b}{6} - \frac{3a}{5} = \frac{89a - 55b}{60}$

20. $\frac{3a - 4b}{7} - \frac{2a - b - c}{3} + \frac{15a - 4c}{12} = \frac{85a - 20b}{84}$

21. $\frac{3a + 2b}{c} - \frac{5bd - 2a - 3d}{4cd} = \frac{12ad + 3bd + 2a + 3d}{4cd}$

22. $\frac{a}{b} + \frac{a - 3b}{cd} + \frac{a^2 - b^2 - ab}{bcd} = \frac{acd - 4b^2 + a^2}{bcd}$

23. $cf + \frac{a^2}{c^5f} - \frac{a - b - c^2}{bc^5f^3} = \frac{bc^6f^4 + a^2bf^2 - a + b + c^2}{bc^5f^3}$

24. $\frac{3a + b + x}{5a} - \frac{2a + b}{3b} + \frac{7a - 2b}{9a} = \frac{47ab - b^2 + 9bx - 30a^2}{45ab}$

25. $\frac{3a^m(a+b)^{m-2}}{c^{m+2}d^{m-3}f^4} - \frac{a^{3m} - 2acd^{4-m}}{c^{m+1}df^n(a+b)^2} - \frac{1}{c^{m-2}f^{n-3}(a+b)^2}$

$$= \frac{3a^mf^{n-4}(a+b)^m - ca^{3m}d^{m-4} + 2ac^2 - c^4f^3d^{m-3}}{c^{m+2}d^{m-3}f^n(a+b)^2}$$

26. $$\frac{(a+x)^{\frac{p}{q}-1}}{3b^2(c+x)^{\frac{m}{n}}} - \frac{b^{\frac{2}{3}}x^2(c+x)^{-\frac{m}{n}}}{(a+x)^{1-\frac{p}{2q}}} = \frac{(a+x)^{\frac{p}{2q}} - 3b^{\frac{2}{3}}x^2}{3b^2(c+x)^{\frac{m}{n}}(a+x)^{1-\frac{p}{2q}}}$$

27. $$\frac{a}{a+z} + \frac{z}{a-z} = \frac{a^2+z^2}{a^2-z^2}$$

28. $$\frac{f+g}{3f-2g} - \frac{5f-2g}{2f-9g} = \frac{9fg-13f^2-13g^2}{6f^2-31fg+18g^2}$$

29. $$\frac{a}{b+x} - \frac{c}{x} + \frac{3c}{4x} + 2b = \frac{8bx^2+(8b^2+4a-c)x-bc}{4bx+4x^2}$$

30. $$\frac{3a+2x}{a+x} - \frac{5a-x}{a-x} + \frac{a}{2x} = \frac{a^3-4a^2x-11ax^2-2x^3}{2x(a^2-x^2)}$$

31. $$\frac{az}{a^2-z^2} - \frac{a-z}{a+z} = \frac{3az-a^2-z^2}{a^2-z^2}$$

32. $$\frac{ac}{a^2-4y^2} + \frac{bd}{ac+2cy} = \frac{ac^2+abd-2bdy}{c(a^2-4y^2)}$$

33. $$\frac{a^3}{(a+b)^3} - \frac{ab}{(a+b)^2} + \frac{b}{a+b} = \frac{a^3+ab^2+b^3}{(a+b)^3}$$

34. $$\frac{a^m}{(a+b)^n} + \frac{a^{m-2}b^r}{(a+b)^{n-1}} - \frac{a^{m-3}b^r}{(a+b)^{n-2}} = \frac{a^m-a^{m-2}b^{r+1}-a^{m-3}b^{r+2}}{(a+b)^n}$$

35. $$\frac{2ax+x^2}{(a-x)^2} - \frac{a^2+5ax}{(a+x)^2} - \frac{x}{a-x} = \frac{2x^4+13a^2x^2-2a^3x-a^4}{(a^2-x^2)^2}$$

36. $$\frac{a-(n+1)a^{n+1}}{1-a} + \frac{a^2(1-a^n)}{(1-a)^2} = \frac{a-(n+1)a^{n+1}+na^{n+2}}{(1-a)^2}$$

37. $$\frac{1}{1-z^2} - \frac{1}{m+1+(m-1)z^2} = \frac{m(1+z^2)}{m(1-z^4)+(1-z^2)^2}$$

38. $$\frac{3}{4(1-x)} + \frac{3}{8(1-x)} + \frac{1}{8(1+x)} - \frac{1-x}{4(1+x^2)} = \frac{1+x+x^2}{1-x-x^4+x^5}$$

39. $$\frac{1+2x}{(3-x)(1+x)} + \frac{7}{(2+x)(1-3x)} + \frac{x}{(1+x)(2+x)} = \frac{23+16x-30x^2-3x^3}{(3-x)(1+x)(2+x)(1-3x)}$$

40. $$\frac{3h}{(h-2x)^2} + \frac{2h+x}{(h+x)(h-2x)} - \frac{5}{h+x} = \frac{20hx-22x^2}{(h+x)(h-2x)^2}$$

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(2) REDUCTION OF THE FACTORS OF FRACTIONS.*

1. $$\frac{ax+x^2}{3bx-cx} = \frac{a+x}{3b-c}$$

2. $$\frac{ac^3-bc^5-c^7}{3bc^2+c^4} = \frac{ac-bc^3-c^5}{3b+c^2}$$

3. $$\frac{21a^3b^2c-9ab^3c^2}{15a^2b^2c+3a^5b^4c^2-12ab^2c} = \frac{7a^2-3bc}{5a+a^4b^2c-4}$$

4. $$\frac{2a^{n+r}b^{m-1}c-4a^rb^{2m-1}c^2d+2a^{r+1}b^mc+6a^{r-1}b^{m-1}c^n}{8a^{r+5}b^{m+2}c^2-2a^{r+3}b^mc+10a^rb^3c^4} = \frac{a^n-2b^mcd+ab+3a^{-1}c^{n-1}}{4a^5b^3c-a^3b+5b^{4-m}c^3}$$

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* The Reduction of the Factors of a Fraction pre-supposes us able to find their common measures. How to find this common measure is shewn in almost all books, and may therefore be supposed known. A similar method applies to literal expressions. It, however, often leads to diffuse calculations. The Solution of Equations, moreover, gives the factors, and may often be employed with advantage. By observing the nature of the Factors of a Fraction, their common measures may generally be discovered, and the reductions made accordingly.

5. $\dfrac{14a^2-7ab}{10ac-5bc}=\dfrac{7a}{5c}$

6. $\dfrac{12a^3x^4+2a^2x^5}{18ab^2x+3b^2x^2}=\dfrac{2a^2x^3}{3b^2}$

7. $\dfrac{6ac+9bc-5c^2}{12adf+18bdf-10cdf}=\dfrac{c}{2df}$

8. $\dfrac{45a^3b^4c+27a^8b^7cd-9a^4b^2d^3}{30a^2b^2c^3d^4+18a^7b^5c^3d^5-6a^3c^2d^7}=\dfrac{3ab^2}{2c^2a^4}$

9. $\dfrac{30a^{3n-1}b^rc^{r+2}-6a^{2n-4}b^3c^rd^{r-1}}{20a^nb^{r-1}c^2d^2-4a^{-3}b^2d^{r+1}}=\dfrac{6a^{2n-1}bc^r}{4d^2}$

10. $\dfrac{5a^2+5ax}{a^2-x^2}=\dfrac{5a}{a-x}$

11. $\dfrac{a^3-x^3}{(a-x)^2}=\dfrac{a^2+ax+x^2}{a-x}$

12. $\dfrac{n^2-2n+1}{n^2-1}=\dfrac{n-1}{n+1}$

13. $\dfrac{a^3+(1+a)ay+y^2}{a^4-y^2}=\dfrac{a+y}{a^2-y}$

14. $\dfrac{ac+bd+ad+bc}{af+2bx+2ax+bf}=\dfrac{c+d}{f+2x}$

15. $\dfrac{6ac+10bc+9ad+15bd}{6c^2+9cd-2c-3d}=\dfrac{3a+5b}{3c-1}$

16. $\dfrac{n^3-2n^2}{n^2-4n+4}=\dfrac{n^2}{n-2}$

17. $\dfrac{x^2+2x-3}{x^2+5x+6}=\dfrac{x-1}{x+2}$

18. $\dfrac{9x^3+53x^2-9x-18}{x^2+11x+30}=\dfrac{9x^2-x-3}{x+5}$

19. $\dfrac{2x^3 + x^2 - 8x + 5}{7x^2 - 12x + 5} = \dfrac{2x^2 + 3x - 5}{7x - 5}$

20. $\dfrac{2x^3 + 3x^2 + x}{x^3 - x^2 - 2x} = \dfrac{2x + 1}{x - 2}$

21. $\dfrac{a^3b^3 + c^3x^3}{a^2b^2 - c^2x^2} = \dfrac{a^2b^2 - abcx + c^2x^2}{ab - cx}$

22. $\dfrac{ax^m - bx^{m+1}}{a^2bx - b^3x^3} = \dfrac{x^{m-1}}{ab + b^2x}$

23. $\dfrac{2x^3 - (3c + d + 2)x^2 + (3c + d)x}{x^4 - x} = \dfrac{2x - 3c - d}{x^2 + x + 1}$

24. $\dfrac{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc}{a^2 - b^2 - c^2 - 2bc} = \dfrac{a + b + c}{a - b - c}$

25. $\dfrac{a^2 - 3ab + ac + 2b^2 - 2bc}{a^2 - b^2 + 2bc - c^2} = \dfrac{a - 2b}{a + b - c}$

26. $\dfrac{(a + b)(a + b + c)(a + b - c)}{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4} =$

$\dfrac{(a+b)(a+b+c)(a+b-c)}{4b^2c^2 - (a^2 - b^2 - c^2)^2} = \dfrac{a+b}{(c+a-b)(b-a+c)}$ *

---

(3) MISCELLANEOUS REDUCTIONS.

1. $\sqrt{ax} + \dfrac{ax}{a - \sqrt{ax}} = \dfrac{a\sqrt{ax}}{a - \sqrt{ax}} = \dfrac{a\sqrt{x}}{\sqrt{a} - \sqrt{x}}$

2. $\dfrac{c\sqrt{x}}{\sqrt{(a+x)}} + \dfrac{d\sqrt{x}}{\sqrt{(a-x)}} + \dfrac{a\sqrt{(ax^3 + x^4)}}{\sqrt{(a^2 - x^2)}} - \sqrt{(a^2 - x^2)}$

$= \dfrac{c\sqrt{(ax - x^2)} + (ax + d)\sqrt{(ax + x^2)} + x^2 - a^2}{\sqrt{(a^2 - x^2)}}$

---

* Upon this reduction several important remarks may be made.

3. $\dfrac{2x^2}{(1-x^2)^{\frac{3}{2}}} - \dfrac{1}{(1-x^2)^{\frac{1}{2}}} = \dfrac{3x^2-1}{(1-x^2)\sqrt{(1-x^2)}}$

4. $\dfrac{ax^3}{(a+x)^{\frac{5}{3}}} + \dfrac{bx^2}{(a+x)^{\frac{4}{3}}} + \dfrac{cx}{(a+x)^{\frac{2}{3}}} = \dfrac{(a+b+c)x^3+(ab+2ac)x^2+a^2cx}{(a+x)^2\sqrt[3]{(a+x)^2}}$

5. $\sqrt{\left(1-\dfrac{f^2}{(f-g)^2}\right)} = \dfrac{\sqrt{(g^2-2fg)}}{f-g}$

6. $\dfrac{a+\sqrt{-b}}{a-\sqrt{-b}} = \dfrac{a^2-b+2a\sqrt{-b}}{a^2+b}$

7. $\dfrac{a+\sqrt{-b}}{a-\sqrt{-b}} + \dfrac{a-\sqrt{-b}}{a+\sqrt{-b}} = \dfrac{2(a^2-b)}{a^2+b}$

8. $\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}} = \dfrac{a+\sqrt{(a^2-x^2)}}{x}$

9. $\dfrac{b}{\sqrt[n]{[a-\sqrt{(a^2-b^n)}]}} = \sqrt[n]{[a+\sqrt{(a^2-b^n)}]}$

10. $\sqrt{(a+\sqrt{b})} \pm \sqrt{(a-\sqrt{b})} = \sqrt{[2a \pm 2\sqrt{(a^2-b)}]}$*

11. $\sqrt{(a+\sqrt{-b})} \pm \sqrt{(a-\sqrt{-b})} = \sqrt{[2a \pm 2\sqrt{(a^2+b)}]}$

12. $\sqrt{\left(\dfrac{abf+c^2}{bc} + \sqrt{\dfrac{4af}{b}}\right)} + \sqrt{\left(\dfrac{abf+c^2}{bc} - \sqrt{\dfrac{4af}{b}}\right)} = \sqrt{\dfrac{4af}{c}}$

---

* This reduction may be effected two ways; (1) by getting the root of $\sqrt{(a+\sqrt{b})}$ and of $\sqrt{(a-\sqrt{b})}$, and taking their form; or (2) by squaring and afterwards taking the square root of the whole expression.

13. $$\frac{\frac{a}{b}+\frac{c}{d}}{\frac{e}{f}+\frac{g}{h}}=\frac{(ad+bc)fh}{(eh+fg)bd}$$

14. $$\frac{\frac{a}{b}+\frac{c}{d}+\frac{e}{f}}{\frac{g}{h}+\frac{i}{k}+\frac{l}{m}}=\frac{(adf+bcf+bde)hkm}{(gkm+him+hkl)bdf}$$

15. $$\frac{\frac{a^3f^3}{b^2c^2}-\frac{a^4f}{bc}+a^2c}{\frac{a^2g}{bc^2d}-\frac{a^6c}{b^2g^2h}+\frac{a^3}{bc}}=\frac{(af^3-a^2bcf+b^2c^3)dg^2h}{bg^3h-a^4c^3d+abcdg^2h}$$

16. $$\frac{\frac{a}{a-b}+\frac{b}{a+b}}{\frac{a}{a-b}-\frac{b}{a+b}}=\frac{a^2+2ab-b^2}{a^2+b^2}$$

17. $$\frac{\frac{c^2}{d^2}-\frac{c^3}{a+b}}{\frac{c^2}{a+b}-\frac{c^4}{dh^2}}=\frac{(a+b-cd^2)h^2}{d^2h^2-(a+b)c^2d}$$

18. $$\frac{1+\frac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}}}{\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}}=\frac{1}{\sqrt{(a^2+x^2)}}$$

19. $$\frac{\sqrt{(1-x)}+\frac{1}{\sqrt{(1+x)}}}{1+\frac{1}{\sqrt{(1-x^2)}}}=\sqrt{(1-x)}$$

20. $$\frac{a^2+ax+x^2}{a^4+a^3x+a^2x^2+ax^3+x^4}=\frac{a^3-x^3}{a^5-x^5}$$

21. $$\frac{a^3-a^2x+ax^2-x^3}{a^5-a^4x+a^3x^2-a^2x^3+ax^4-x^5}=\frac{a^4-x^4}{a^6-x^6}$$

22. $\dfrac{a^2 + 2ax + 4x^2}{a^3 - 2a^2x + 4ax^2 - 8x^3} = \dfrac{a^3 + 8x^3}{a^4 - 16x^4}$

23. $9\sqrt{(6\sqrt{28})} + 3\sqrt{(12\sqrt{7})} - 8\sqrt{(4\sqrt{63})} = 8\sqrt[4]{63}$

24. $3\sqrt{(40\sqrt{12})} + 2\sqrt{(5\sqrt{48})} - 4\sqrt{(15\sqrt{27})} = 4\sqrt[4]{75}$

25. $4\sqrt[3]{(6\sqrt{32})} + \sqrt[3]{(9\sqrt{162})} + 2\sqrt[3]{(75\sqrt{50})} = 21\sqrt[6]{18}$

26. $5\sqrt[3]{(4\sqrt[3]{192})} + 7\sqrt[3]{(18\sqrt[3]{81})} = 31\sqrt[9]{24}$

27. $3\sqrt[3]{(8 + 16\sqrt{5})} - 2\sqrt[3]{(1 + \sqrt{20})} = 4\sqrt[3]{(1 + 2\sqrt{5})}$

28. $3\sqrt[3]{(54 - 36\sqrt{27})} - \sqrt[3]{(16 - 16\sqrt{12})} = 7\sqrt[3]{(2 - 4\sqrt{3})}$

29. $(aa' + bb')^2 + (ab' - ba')^2 = (a^2 + b^2)(a'^2 + b'^2)$*

30. $(aa' + bb')^2 + (ab' - ba')^2 + a'^2c^2 + b'^2c^2 = (a^2 + b^2 + c^2)(a'^2 + b'^2)$

31. $(aa' + bb' + cc')^2 + (ab' - ba')^2 + (ac' - ca')^2 + (bc' - cb')^2 = (a^2 + b^2 + c^2)(a'^2 + b'^2 + c'^2)$

32. $(aa' + bb' + cc' + dd')^2 + (ab' - ba' + cd' - dc')^2 + (ac' - bd' - ca' + db')^2 + (ad' + bc' - cb' - da')^2 = (a^2 + b^2 + c^2 + d^2)(a'^2 + b'^2 + c'^2 + d'^2)$

33. $(a^2 \pm Ab^2)(a'^2 \pm Ab'^2) = (aa' \pm Abb')^2 + A(ab' \mp ba')^2$†

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* For the sake of symmetry we often distinguish letters by an accent or dash from those which have the same rank in the alphabet. For instance, a', b', c', &c. are different from a, b, c, &c.

† The forms 29, 30, 31, 32, 33, solve certain Problems in Indeterminate Analysis which will be seen hereafter. Their accuracy will be evinced by the actual developement of the squares and products. The Reader may derive some advantage from this mode of solution.

34. $(ab' - ba')(ab'' - ba'') + (bc' - cb')(bc'' - cb'') + (ca' - ac')(ca'' - ac'') = (a^2 + b^2 + c^2)(a'a'' + b'b'' + c'c'') - (aa' + bb' + cc')(aa'' + bb'' + cc'')$

VIII. LOGARITHMS.

WHAT is the Log. of a Number? What its Cardinal Number or Base? What are we to understand, when, for example, it is said, "for the Base a of a log. let $N = 6{\cdot}67$?" What is a System of Logarithms? And what is the particular System which is called Brigg's System? How can the others be represented by words under the three following principal forms? And how demonstrated? Can 1 be properly assumed as a Base of a System? What is the Logarithm of 1. When the Base is > 1, the log. of a number > 1 is positive, and that of a number < 1 is negative. But how is it when the Base is < 1? A few logarithms only are Integers, the others being partly Integer and partly Fractional (indeed in Brigg's System always a Mixed Number) and do not admit of exact representation. What is this Integer part called? and what the Fractional one? In Brigg's System how do you characterise the numbers, which are between 10^n and 10^{n+1}, and between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$?

When the differences of the numbers are small compared with the numbers themselves, then the differences of the logarithms are nearly as the differences of the Numbers. Analysis proves this. What is the use of the proportional parts, as given in the larger Tables of Logarithms?

(1) PRINCIPAL FORMS.

1. $\log AB = \log A + \log B$
2. $\log \frac{A}{B} = \log A - \log B$
3. $\log A^n = n \log A$

Observe. In (3) n may be positive, negative, Integer, or Fractional.

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(2) APPLICATION TO THE FINDING LOGARITHMS OF PRODUCTS, QUOTIENTS, POWERS, AND ROOTS.

(a) *For General or Literal Expressions.*

1. $\log abcd = \log a + \log b + \log c + \log d$
2. $\log \frac{fg}{cd} = \log f + \log g - \log c - \log d$
3. $\log a^m b^n c^p = m \log a + n \log b + p \log c$
4. $\log \frac{a^m b^{-n}}{c^p d^q} = m \log a - n \log b - p \log c - q \log d$
5. $\log a^{\frac{m}{n}} b^{-\frac{p}{q}} c = \frac{m}{n} \log a - \frac{p}{q} \log b + \log c$
6. $\log \sqrt[n]{a^m b^{-n} c^{\frac{p}{q}}} = \frac{m}{n} \log a - \log b + \frac{p}{nq} \log c$
7. $\log \frac{a\sqrt[n]{c^m}}{b\sqrt{d}} = \log a + \frac{m}{n} \log c - \log b - \frac{1}{2} \log d$

8. $\log \dfrac{(a+b)^n c^m}{(c+d)\sqrt[4]{d^3}} = n \log (a+b) + m \log c - \log (c+d) - \frac{3}{4} \log d$

9. $\log \dfrac{1}{(a+b^n)^m} = -m \log (a+b^n)$

10. $\log \dfrac{1}{\sqrt[n]{(a+b)}} = -\dfrac{1}{n} \log (a+b)$

11. $\log \sqrt[m]{(a^2-x^2)} = \dfrac{1}{m} \log (a^2-x^2) = \dfrac{1}{m} \log (a+x) + \dfrac{1}{m} \log (a-x)$

12. $x \log a = \log a^x$

13. $n \log a + m \log b - p \log c = \log \dfrac{a^n b^m}{c^p}$

14. $n \log (a+y) + \log c - m \log (a-y) = \log \dfrac{c\,(a+y)^n}{(a-y)^m}$

15. $\dfrac{1}{n} \log (2a+3b) - \frac{2}{3} \log c = \log \dfrac{\sqrt[n]{(2a+3b)}}{\sqrt[3]{c^2}}$

(b) For Numerical Expressions according to Brigg's System.

1. $\log (93 \times 3514) = 5{\cdot}5142847$
2. $\log (1225 \times 387) = 5{\cdot}6758471$
3. $\log (628 \times 493) = 5.4908066$
4. $\log (3748 \times 1752 \times 4065) = 10{\cdot}4263942$
5. $\log \frac{5}{4} = 0{\cdot}0969100$
6. $\log \frac{39}{7} = 0{\cdot}7459666$
7. $\log \frac{14}{3} = 0{\cdot}6690068$
8. $\log 15\frac{3}{4} = 1{\cdot}1972806$
9. $\log 7\frac{4}{13} = 0{\cdot}8637803$

10. $\log 367\frac{5}{8} = 2{\cdot}5654050$
11. $\log 187\frac{9}{11} = 2{\cdot}2737376$
12. $\log \frac{2}{3} = 0{\cdot}8239087 - 1$
13. $\log \frac{5}{8} = 0{\cdot}7958800 - 1$
14. $\log \frac{1}{9} = 0{\cdot}0457575 - 1$
15. $\log \frac{1}{26} = 0{\cdot}5850267 - 2$
16. $\log \frac{6}{37} = 0{\cdot}2099495 - 1$
17. $\log \frac{18}{1267} = 0{\cdot}1524959 - 2$
18. $\log \frac{557}{693} = 0{\cdot}7106834 - 1$
19. $\log \frac{1}{3432} = 0{\cdot}2650402 - 4$
20. $\log \frac{8}{9243} = 0{\cdot}9372770 - 4$
21. $\log \frac{3542}{7963} = 0{\cdot}6480628 - 1$
22. $\log 3{\cdot}5 = 0{\cdot}5440680$
23. $\log 12{\cdot}63 = 1{\cdot}1014034$
24. $\log 15{\cdot}432 = 1{\cdot}1884222$
25. $\log 7348{\cdot}4 = 3{\cdot}8661928$
26. $\log 1{\cdot}3567 = 0{\cdot}1324838$
27. $\log 0{\cdot}7 = 0{\cdot}8450980 - 1$
28. $\log 0{\cdot}036 = 0{\cdot}5563025 - 2$
29. $\log 0{\cdot}0065 = 0{\cdot}8129134 - 3$
30. $\log 0{\cdot}0039953 = 0{\cdot}6015494 - 3$
31. $\log 0{\cdot}0005637 = 0{\cdot}7510480 - 4$
32. $\log \dfrac{319 \times 765}{138} = 3{\cdot}2475730$
33. $\log \dfrac{213 \times 7{\cdot}655}{3145 \times 718} = 0{\cdot}8585798 - 4$

34. $\log \dfrac{3{\cdot}5347 \times 2{\cdot}685}{137{\cdot}65 \times 5944} = 0{\cdot}0644419 - 5$

35. $\log \dfrac{47 \times 0{\cdot}653 \times 12\frac{5}{6}}{3576 \times 1520} = 0{\cdot}8601095 - 5$

36. $\log \dfrac{0{\cdot}765 \times 0{\cdot}0018}{31457 \times 567\frac{5}{12}} = 0{\cdot}8873146 - 11$

37. $\log \dfrac{0{\cdot}018594 \times 76\frac{13}{13}}{7654{\cdot}3 \times 794} = 0{\cdot}3686643 - 6$

38. $\log 3^{15} = 7{\cdot}1568188$*)

39. $\log 5^{27} = 18{\cdot}8721901$

40. $\log 16^{20} = 24{\cdot}0823997$

41. $\log (\frac{7}{3})^{14} = 5{\cdot}1516750$

42. $\log (\frac{15}{13})^{16} = 0{\cdot}9943665$

43. $\log (\frac{12}{5})^{32} = 12{\cdot}1667597$

44. $\log (\frac{3}{4})^{30} = 0{\cdot}2518379 - 4$

45. $\log (\frac{357}{1432})^{12} = 0{\cdot}7607024 - 8$

46. $\log (\frac{1}{14})^{125} = 0{\cdot}7339955 - 144$

47. $\log [(14{\cdot}418)^9 \times (3{\cdot}71)^6] = 13{\cdot}8463886$

48. $\log [(0{\cdot}0534)^3 \times (\frac{32768}{3875})^{10}] = 5{\cdot}4544061$

49. $\log \dfrac{(0{\cdot}5936)^{20} \times 386}{(0{\cdot}076)^{23}} = 23{\cdot}7977525$

50. $\log \sqrt{5} = 0{\cdot}3494850$

51. $\log \sqrt{73567} = 2{\cdot}4333415$

52. $\log \sqrt[3]{135} = 0{\cdot}7101112$

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* To obtain the logarithms of rather high powers, true to 7 decimal places, care must be had to use logarithms of *more* than 7 decimal places.

53. $\log \sqrt[3]{15276} = 0{\cdot}5230012$

54. $\log \sqrt[5]{35107} = 0{\cdot}9090787$

55. $\log \sqrt[100]{13} = 0{\cdot}0111394$

56. $\log \sqrt[7]{\frac{15}{4}} = 0{\cdot}0820045$

57. $\log \sqrt[5]{\frac{4}{9}} = 0{\cdot}9295635 - 1$

58. $\log \sqrt[16]{\frac{3587}{20591}} = 0{\cdot}9525632 - 1$

59. $\log \sqrt[35]{\frac{803}{91056}} = 0{\cdot}9412973 - 1$

60. $\log \sqrt[17]{(954)^{12}} = 2{\cdot}1032106$

61. $\log \sqrt[11]{(\frac{12}{7})^{28}} = 0{\cdot}5958482$

62. $\log \sqrt[13]{(\frac{547}{935})^{207}} = 0{\cdot}2927210 - 4$

63. $\log \sqrt[14]{(\frac{1}{3})^{187}} = 0{\cdot}6270232 - 7$

64. $\log \sqrt[16]{(\frac{14}{207})^{715}} = 0{\cdot}7828746 - 58$

65. $\log \sqrt[80]{0{\cdot}00534} = 0{\cdot}9715943 - 1$

66. $\log \sqrt[540]{0{\cdot}00007} = 0{\cdot}9923057 - 1$

67. $\log \sqrt[12]{(0{\cdot}34576)^7} = 0{\cdot}7309519 - 1$

68 $\log \sqrt[32]{(356{\cdot}27)^{11}} = 0{\cdot}8771741$

69. $\log \sqrt[5]{\dfrac{0{\cdot}365 \times \sqrt{2}}{788}} = 0{\cdot}3632563 - 1$

70. $\log \sqrt[10]{\dfrac{78563\sqrt[3]{\frac{5}{3}}}{15\sqrt[4]{0{\cdot}2}}} = 0{\cdot}3967819$

71. $\log \sqrt[9]{\dfrac{347\sqrt[7]{0{\cdot}0073}}{126\sqrt[3]{\frac{4}{9}}}} = 0{\cdot}0280126$

(3) THE USE OF THE PROPORTIONAL PARTS IN LOGARITHMS.

(a) The Logarithms of such numbers as exceed those in the Tables.

1. log 1851273 = 6·2674705
2. log 14459809 = 7·1601626
3. log 10134761 = 7·0058135
4. log 7095137 = 6·8509608
5. log 506860900 = 8·7048888
6. log 3·614699 = 0·5580721
7. log 84·827567 = 1·9285370
8. log 211447·39 = 5·3252023
9. log 0·0013514133 = 0·1307882 − 3
10. log 0·0003599547 = 0·5562478 − 4
11. log $75907\frac{1}{8}$ = 4·8802825
12. log $32116\frac{7}{9}$ = 4·5067320
13. log $2528811\frac{1}{4}$ = 6·4029164
14. log $522076\frac{2}{13}$ = 5·7177339

(b) Determination of the numbers which belong to such Logarithms as are not found accurately in the Table.

1. num. log 1·0742664 = 11·86496....
2. num. log 3·5947835 = 3933·538....
3. num. log 0·7813427 = 6·044254....
4. num. log 2·0037683 = 100·8714....
5. num. log 4·0005673 = 10013·07....

6. num. log 5·6165834 = 413602·7....
7. num. log 3·7694480 = 5880·956....
8. num. log 9·2307611 = 1·701222....
9. num. log 4·2923065 = 19602·27....
10. num. log 6·1785400 = 1508481·....

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(4) ACTUAL CALCULATION OF SOME NUMERICAL EXPRESSIONS, BY MEANS OF LOGARITHMS.

1. $\sqrt[7]{8} = 1{\cdot}345900....$
2. $\sqrt[4]{35246} = 13{\cdot}70179....$
3. $\sqrt[12]{567348} = 3{\cdot}016389....$
4. $\sqrt[6]{235{\cdot}78} = 2{\cdot}485522....$
5. $\sqrt[5]{\frac{13}{16}} = 0{\cdot}959322....$
6. $\sqrt[7]{\frac{1171}{343}} = 1{\cdot}190747....$
7. $\sqrt[3]{17705\frac{2}{9}} = 26{\cdot}06356....$
8. $\sqrt[9]{1350\frac{7}{8}} = 2{\cdot}227645....$
9. $\sqrt[8]{172\frac{5}{8}} = 1{\cdot}904159....$
10. $\sqrt[13]{\frac{3348}{569}} = 1{\cdot}146055....$
11. $(\frac{9}{8})^{21} = 11{\cdot}86322....$
12. $(2\frac{5}{6})^{9} = 11767{\cdot}35....$
13. $(\frac{643}{637})^{123} = 3{\cdot}168104....$
14. $(317\frac{3}{4})^{0{\cdot}6} = 31{\cdot}71402....$
15. $(\frac{167}{33})^{0{\cdot}32} = 1{\cdot}443779....$

16. $(\frac{5}{7})^{0.0537} = 0.982093....$

17. $\dfrac{(991.767)^5 \times 12.34}{(20.358 \times 10.1575)^6} = 151.4369....$

18. $\dfrac{(52072)^{13} \times \sqrt{(0.000734)^9}}{(255608)^8} = 8930.834...$

19. $\left(\dfrac{42666}{1147}\right)^{12} \times \left(\dfrac{765}{19432}\right)^{10} = 62756.88....$

20. $\sqrt[5]{(\frac{7}{3}\sqrt[4]{6})} = 1.295695....$

21. $\sqrt[3]{(0.26 \cdot \sqrt{\frac{2}{3}})} = 0.596544....$

22. $\sqrt[5]{\dfrac{3425\sqrt[7]{136}}{0.00034}} = 28.94639....$

23. $253\sqrt[3]{\dfrac{716.5}{\sqrt{2}}} = 2016.914....$

24. $\sqrt[4]{\dfrac{132 \times (7.356)^9}{\sqrt{(3.25)^5}}} = 144.5972....$

25. $\dfrac{\sqrt[7]{(466871)^6} \times \sqrt[9]{(3576)^{16}}}{996003\sqrt{0.0071}} = 1788845....$

26. $\sqrt[8]{(21 + \sqrt[6]{19})} = 1.476875....$

27. $\sqrt[3]{(5.03 + \sqrt[5]{0.2})} = 1.792020....$

28. $\sqrt[5]{(9.921 - 3\sqrt{5.02})} = 1.261866....$

29. $\dfrac{\sqrt[16]{43 + 5\sqrt[3]{278}}}{\sqrt[5]{17}} = 1.264848....$

.....

The following easy expressions may also be noticed with advantage:

(1.) Let A, B, denote two systems of Logarithms, whose respective Bases are a, b. Let also x, y be the logarithms of the same number k in these systems. Then $y : x :: \log. a : \log. b$, the two last logarithms belonging to any third system whatever.

(2.) Hence the log. of a number k in the system B, is found by dividing the log. of k in the system A by the log. of b in that system.

(3.) Hence also the logarithms of the same numbers in two different systems have a constant ratio.

(4.) If, therefore, we know the log. of one and the same number in two different systems, then a number may be found whereby we may multiply the logarithms of all numbers in one system in order to find their logarithms in the other. This number is termed the *Modulus*. In the common acceptation of the word, it means that Modulus which serves to transform the log. in the Hyperbolical or Natural system, whose Base is 2·718281...., to that of any other system.

XI. PERMUTATIONS, COMBINATIONS, AND VARIATIONS.

WHAT is a Permutation? What a Combination? What a Variation? And in what respects do they differ?*

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* *Permutations* means the transpositions that can take place between any number of things, *all* of them being taken together. By *Variations* we understand the transpositions, when the things are *not all* taken together. Most authors consider both cases as one, and term them *Permutations*.

The theory of Combinations, considered as an independent science, is not limited to the three heads just enumerated. In its extended sense it will not be here considered.

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(1) PERMUTATIONS.

(a) An actual view of the Permutations of given things or Elements.

1. Perm. $(abc) =$

abc
acb
bac
bca
cab
cba

2. Perm. $(aabb) =$

aabb
abab
abba
baab
baba
bbaa

3. Perm. $(abcd) =$

abcd	*cabd*
abdc	*cadb*
acbd	*cbad*
acdb	*cbda*
adbc	*cdab*
adcb	*cdba*
bacd	*dabc*
badc	*dacb*
bcad	*dbac*
bcda	*dbca*
bdac	*dcab*
bdca	*dcba*

4. Perm. $(abbc) =$

abbc
abcb
acbb
babc
bacb
bbac
bbca
bcab
bcba
cabb
cbab
cbba

5. Perm. (*aabbc*) =

aabbc	*babca*
aabcb	*bacab*
aacbb	*bacba*
ababc	*bbaac*
abacb	*bbaca*
abbac	*bbcaa*
abbca	*bcaab*
abcab	*bcaba*
abcba	*bcbaa*
acabb	*caabb*
acbab	*cabab*
acbba	*cabba*
baabc	*cbaab*
baacb	*cbaba*
babac	*cbbaa*

6. Perm. (*aaaabb*) =

aaaabb
aaabab
aaabba
aabaab
aababa
aabbaa
abaaab
abaaba
ababaa
abbaaa
baaaab
baaaba
baabaa
babaaa
bbaaaa

7. Perm. (*aabcd*) =

aabcd	*acabd*	*baacd*	*caabd*	*daabc*
aabdc	*acadb*	*baadc*	*caadb*	*daacb*
aacbd	*acbad*	*bacad*	*cabad*	*dabac*
aacdb	*acbda*	*bacda*	*cabda*	*dabca*
aadbc	*acdab*	*badac*	*cadab*	*dacab*
aadcb	*acdba*	*badca*	*cadba*	*dacba*
abacd	*adabc*	*bcaad*	*cbaad*	*dbaac*
abadc	*adacb*	*bcada*	*cbada*	*dbaca*
abcad	*adbac*	*bcdaa*	*cbdaa*	*dbcaa*
abcda	*adbca*	*bdaac*	*cdaab*	*dcaab*
abdac	*adcab*	*bdaca*	*cdaba*	*dcaba*
abdca	*adcba*	*bdcaa*	*cdbaa*	*dcbaa*

8. Perm. (*aabbcc*) =

aabbcc	*acabbc*	*bacabc*	*bcbaac*	*cbaabc*
aabcbc	*acabcb*	*bacacb*	*bcbaca*	*cbaacb*
aabccb	*acacbb*	*bacbac*	*bcbcaa*	*cbabac*
aacbbc	*acbabc*	*bacbca*	*bccaab*	*cbabca*
aacbcb	*acbacb*	*baccab*	*bccaba*	*cbacab*
aaccbb	*acbbac*	*baccba*	*bccbaa*	*cbacba*
ababcc	*acbbca*	*bbaacc*	*caabbc*	*cbbaac*
abacbc	*acbcab*	*bbacac*	*caabcb*	*cbbaca*
abaccb	*acbcba*	*bbacca*	*caacbb*	*cbbcaa*
abbacc	*accabb*	*bbcaac*	*cababc*	*cbcaab*
abbcac	*accbab*	*bbcaca*	*cabacb*	*cbcaba*
abbcca	*accbba*	*bbccaa*	*cabbac*	*cbcbaa*
abcabc	*baabcc*	*bcaabc*	*cabbca*	*ccaabb*
abcacb	*baacbc*	*bcaacb*	*cabcab*	*ccabab*
abcbac	*baaccb*	*bcabac*	*cabcba*	*ccabba*
abcbca	*babacc*	*bcabca*	*cacabb*	*ccbaab*
abccab	*babcac*	*bcacab*	*cacbab*	*ccbaba*
abccba	*babcca*	*bcacba*	*cacbba*	*ccbbaa*

(*b*) *The number of Transpositions.**

FORMULA.

(I.) The number of Transpositions of N different things is $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \ldots (N-1)N$.

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* The Exponents which occur here merely as repeating ones, denote how often the letter to which they belong is repeated. n. P. signifies *numerus Permutationum* (number of Transpositions or Permutations).—N. B. Transpositions merely infers the Permutations of the things taken all together.

(II.) If l of these things be of one kind, m of another, n, of a third, and so on; then if $l+m+n+p+$, &c. be put $= N$, the number of Transpositions of the factors of the form $a^l b^m c^n d^p$.....

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdots\cdots\cdots\cdots (N-1)N}{1\cdot 2\cdot 3\cdots l \times 1\cdot 2\cdot 3\cdots m \times 1\cdot 2\cdot 3\cdots n \times 1\cdot 2\cdot 3\cdots p \times \cdots}$$

$$= \frac{(l+1)(l+2)(l+3)\cdots\cdots\cdots\cdots (N-1)N}{1\cdot 2\cdot 3\cdots m \times 1\cdot 2\cdot 3\cdots n \times 1\cdot 2\cdot 3\cdots p \times \cdots}$$

$$= \frac{(m+1)(m+2)(m+3)\cdots\cdots\cdots (N-1)N}{1\cdot 2\cdot 3\cdots l \times 1\cdot 2\cdot 3\cdots n \times 1\cdot 2\cdot 3\cdots p \times \cdots}$$

$$= \frac{(n+1)(n+2)(n+3)\cdots\cdots\cdots (N-1)N}{1\cdot 2\cdot 3\cdots l \times 1\cdot 2\cdot 3\cdots m \times 1\cdot 2\cdot 3\cdots p \times \cdots}$$

where the products $1\cdot 2\cdot 3\cdots l$, $1\cdot 2\cdot 3\cdots m$, $1\cdot 2\cdot 3\cdots n$, $1\cdot 2\cdot 3\cdots p$, &c. are the same in number as the different kinds of things.

Examples.

1. n. P. $(a) = 1$
2. n. P. $(ab) = 2$
3. n. P. $(abc) = 6$
4. n. P. $(abcd) = 24$
5. n. P. $(abcde) = 120$
6. n. P. $(abcdef) = 720$
7. n. P. $(a^2bc) = 12$
8. n. P. $(a^2b^2) = 6$
9. n. P. $(a^3b^5c^2) = 2520$
10. n. P. $(a^2b^4cd) = 840$

11. n. P. $(a^2b^2c^2) = 90$

12. n. P. $(a^3b^7c^4d) = 1801800$

13. n. P. $(a^2b^3c^8d^4) = 30630600$

14. n. P. $(a^2b^{10}cd^2e^3) = 73513440$

15. n. P. $(a^5b^5c^4d^2) = 30270240$

16. n. P. $(a^{13}b^2c^5d^3e^2) = 864913896000$

17. n. P. $(a^3b^7c^4d^2e^3f^5) = 593676898214400$

18. n. P. $(a^m) = 1$

19. n. P. $(a^{m-1}b) = m$

20. n. P. $(a^{m-2}b^2) = \frac{m(m-1)}{1 \cdot 2}$

21. n. P. $(a^{m-3}b^3) = \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}$

22. n. P. $(a^{m-4}b^4) = \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4}$

23. n. P. $(a^{m-5}b^5) = \frac{m(m-1)(m-2)(m-3)(m-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$

---

(2) COMBINATIONS.

(a) Combinations with Repetitions.

1. Comb. $(a, b, c, d,)$ of the second Class.

   aa, ab, ac, ad, bb, bc, bd, cc, cd, dd

2. Comb. (a, b, c, d, e) of second Class.

   aa, ab, ac, ad, ae, bb, bc, bd, be, cc, cd, ce, dd, de, ee.

3. Comb. (*a*, *b*, *c*)

Third Class.

aaa
aab
aac
abb
abc
acc
bbb
bbc
bcc
ccc

4. Comb. (*a*, *b*, *c*, *d*)

Third Class.

aaa	*bbb*
aab	*bbc*
aac	*bbd*
aad	*bcc*
abb	*bcd*
abc	*bdd*
abd	*ccc*
acc	*ccd*
acd	*cdd*
add	*ddd*

5. Comb. (*a*, *b*, *c*, *d*, *e*)

Third Class.

aaa	*add*	*bee*
aab	*ade*	*ccc*
aac	*aee*	*ccd*
aad	*bbb*	*cce*
aae	*bbc*	*cdd*
abb	*bbd*	*cde*
abc	*bbe*	*cee*
abd	*bcc*	*ddd*
abe	*bcd*	*dde*
acc	*bce*	*dee*
acd	*bdd*	*eee*
ace	*bde*	

6. Comb. (*a*, *b*, *c*)

Fourth Class.

aaaa
aaab
aaac
aabb
aabc
aacc
abbb
abbc
abcc
accc
bbbb
bbbc
bbcc
bccc
cccc

7. Comb. (*a*, *b*, *c*, *d*)
Fourth Class.

aaaa	*abbd*	*bbcd*
aaab	*abcc*	*bbdd*
aaac	*abcd*	*bccc*
aaad	*abdd*	*bccd*
aabb	*accc*	*bcdd*
aabc	*accd*	*bddd*
aabd	*acdd*	*cccc*
aacc	*addd*	*cccd*
aacd	*bbbb*	*ccdd*
aadd	*bbbc*	*cddd*
abbb	*bbbd*	*dddd*
abbc	*bbcc*	

8. Comb. (*a*, *b*, *c*, *d*, *e*)
Fourth Class.

aaaa	*abbe*	*bbbc*	*bdee*
aaab	*abcc*	*bbbd*	*beee*
aaac	*abcd*	*bbbe*	*cccc*
aaad	*abce*	*bbcc*	*cccd*
aaae	*abdd*	*bbcd*	*ccce*
aabb	*abde*	*bbce*	*ccdd*
aabc	*abee*	*bbdd*	*ccde*
aabd	*accc*	*bbde*	*ccee*
aabe	*accd*	*bbee*	*cddd*
aacc	*acce*	*bccc*	*cdde*
aacd	*acdd*	*bccd*	*cdee*
aace	*acde*	*bcce*	*ceee*
aadd	*acee*	*bcdd*	*dddd*
aade	*addd*	*bcde*	*ddde*
aaee	*adde*	*bcee*	*ddee*
abbb	*adee*	*bddd*	*deee*
abbc	*acec*	*bdde*	*eeee*
abbd	*bbbb*		

9. Comb. (*a*, *b*, *c*)

Fifth Class.

aaaaa	*abbbc*
aaaab	*abbcc*
aaaac	*abccc*
aaabb	*acccc*
aaabc	*bbbbb*
aaacc	*bbbbc*
aabbb	*bbbcc*
aabbc	*bbccc*
aabcc	*bcccc*
aaccc	*ccccc*
abbbb	

10. Comb. (*a*, *b*, *c*, *d*)

Fifth Class.

aaaaa	*aabcd*	*abcdd*	*bbccd*
aaaab	*aabdd*	*abddd*	*bbcdd*
aaaac	*aaccc*	*acccc*	*bbddd*
aaaad	*aaccd*	*acccd*	*bcccc*
aaabb	*aacdd*	*accdd*	*bcccd*
aaabc	*aaddd*	*acddd*	*bccdd*
aaabd	*abbbb*	*adddd*	*bcddd*
aaacc	*abbbc*	*bbbbb*	*bdddd*
aaacd	*abbbd*	*bbbbc*	*ccccc*
aaadd	*abbcc*	*bbbbd*	*ccccd*
aabbb	*abbcd*	*bbbcc*	*cccdd*
aabbc	*abbdd*	*bbbcd*	*ccddd*
aabbd	*abccc*	*bbbdd*	*cdddd*
aabcc	*abccd*	*bbccc*	*ddddd*

The number of Combinations with repetitions for n things is:

For the 1st Class $= n$

——— 2nd Class $= \frac{n(n+1)}{1 \cdot 2}$

——— 3rd Class $= \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$

——— 4th Class $= \frac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4}$

.....

——— mth Class $= \frac{n(n+1)(n+2)\ldots.(n+m-1)}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots m}$

(b) Combinations without Repetitions.

1. Comb. $(a, b, c, d, e, f, g, h, i)$

Second Class.

ab	*ah*	*bg*	*cg*	*dh*	*fg*
ac	*ai*	*bh*	*ch*	*di*	*fh*
ad	*bc*	*bi*	*ci*	*ef*	*fi*
ae	*bd*	*cd*	*de*	*eg*	*gh*
af	*be*	*ce*	*df*	*eh*	*gi*
ag	*bf*	*cf*	*dg*	*ei*	*hi*

2. Comb. (a, b, c, d, e, f)

Third Class.

abc	*acd*	*adf*	*bcf*	*cde*
abd	*ace*	*aef*	*bde*	*cdf*
abe	*acf*	*bcd*	*bdf*	*cef*
abf	*ade*	*bce*	*bef*	*def*

3. Comb. (a, b, c, d, e, f, g, h)

Third Class.

abc	*adf*	*bcf*	*bfh*	*cgh*
abd	*adg*	*bcg*	*bgh*	*def*
abe	*adh*	*bch*	*cde*	*deg*
abf	*aef*	*bde*	*cdf*	*deh*
abg	*aeg*	*bdf*	*cdg*	*dfg*
abh	*aeh*	*bdg*	*cdh*	*dfh*
acd	*afg*	*bdh*	*cef*	*dgh*
ace	*afh*	*bef*	*ceg*	*efg*
acf	*agh*	*beg*	*ceh*	*efh*
acg	*bcd*	*beh*	*cfg*	*egh*
ach	*bce*	*bfg*	*cfh*	*fgh*
ade				

4. Comb. (a, b, c, d, e, f, g)

Fourth Class.

abcd	*abfg*	*adfg*	*bdeg*
abce	*acde*	*aefg*	*bdfg*
abcf	*acdf*	*bcde*	*befg*
abcg	*acdg*	*bcdf*	*cdef*
abde	*acef*	*bcdg*	*cdeg*
abdf	*aceg*	*bcef*	*cdfg*
abdg	*acfg*	*bceg*	*cefg*
abef	*adef*	*bcfg*	*defg*
abeg	*adeg*	*bdef*	

5. Comb. (a, b, c, d, e, f, g, h)

Fifth Class.

abcde	*abdfh*	*acegh*	*bcefh*
abcdf	*abdgh*	*acfgh*	*bcegh*
abcdg	*abefg*	*adefg*	*bcfgh*
abcdh	*abefh*	*adefh*	*bdefg*
abcef	*abegh*	*adegh*	*bdefh*
abceg	*abfgh*	*adfgh*	*bdegh*
abceh	*acdef*	*aefgh*	*bdfgh*
abcfg	*acdeg*	*bcdef*	*befgh*
abcfh	*acdeh*	*bcdeg*	*cdefg*
abcgh	*acdfg*	*bcdeh*	*cdefh*
abdef	*acdfh*	*bcdfg*	*cdegh*
abdeg	*acdgh*	*bcdfh*	*cdfgh*
abdeh	*acefg*	*bcdgh*	*cefgh*
abdfg	*acefh*	*bcefg*	*defgh*

The Number of Combinations without Repetitions for n things is:

For the 1st Class $= n$

——— 2nd Class $= \frac{n(n-1)}{1 \cdot 2}$

——— 3rd Class $= \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$

——— 4th Class $= \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$

.....

——— mth Class $= \frac{n(n-1)(n-2) \ldots . (n-m+1)}{1 \cdot 2 \cdot 3 \ldots . . . . . m}$

(3) VARIATIONS.

(a) Variations with Repetitions.

1. Var. (*a*, *b*, *c*, *d*, *e*, *f*)

Second Class.

aa	*ba*	*ca*	*da*	*ea*	*fa*
ab	*bb*	*cb*	*db*	*eb*	*fb*
ac	*bc*	*cc*	*dc*	*ec*	*fc*
ad	*bd*	*cd*	*dd*	*ed*	*fd*
ae	*be*	*ce*	*de*	*ee*	*fe*
af	*bf*	*cf*	*df*	*ef*	*ff*

2. Var. (*a*, *b*, *c*, *d*)

Third Class.

aaa	*adb*	*bcc*	*cbd*	*dba*
aab	*adc*	*bcd*	*cca*	*dbb*
aac	*add*	*bda*	*ccb*	*dbc*
aad	*baa*	*bdb*	*ccc*	*dbd*
aba	*bab*	*bdc*	*ccd*	*dca*
abb	*bac*	*bdd*	*cda*	*dcb*
abc	*bad*	*caa*	*cdb*	*dcc*
abd	*bba*	*cab*	*cdc*	*dcd*
aca	*bbb*	*cac*	*cdd*	*dda*
acb	*bbc*	*cad*	*daa*	*ddb*
acc	*bbd*	*cba*	*dab*	*ddc*
acd	*bca*	*cbb*	*dac*	*ddd*
ada	*bcb*	*cbc*	*dad*	

3. Var. (a, b, c)

Fourth Class.

aaaa	*abcc*	*baca*	*bcbb*	*cbac*
aaab	*acaa*	*bacb*	*bcbc*	*cbba*
aaac	*acab*	*bacc*	*bcca*	*cbbb*
aaba	*acac*	*bbaa*	*bccb*	*cbbc*
aabb	*acba*	*bbab*	*bccc*	*cbca*
aabc	*acbb*	*bbac*	*caaa*	*cbcb*
aaca	*acbc*	*bbba*	*caab*	*cbcc*
aacb	*acca*	*bbbb*	*caac*	*ccaa*
aacc	*accb*	*bbbc*	*caba*	*ccab*
abaa	*accc*	*bbca*	*cabb*	*ccac*
abab	*baaa*	*bbcb*	*cabc*	*ccba*
abac	*baab*	*bbcc*	*caca*	*ccbb*
abba	*baac*	*bcaa*	*cacb*	*ccbc*
abbb	*baba*	*bcab*	*cacc*	*ccca*
abbc	*babb*	*bcac*	*cbaa*	*cccb*
abca	*babc*	*bcba*	*cbab*	*cccc*
abcb				

The number of Variations with repetitions for n things and the mth Class is n^m.

(b) Variations without Repetitions.

Var. (*a*, *b*, *c*, *d*, *e*)

Fourth Class.

abcd	*aecb*	*bdea*	*cdab*	*dbca*	*eadb*
abce	*aecd*	*bdec*	*cdae*	*dbce*	*eadc*
abdc	*aedb*	*beac*	*cdba*	*dbea*	*ebac*
abde	*aedc*	*bead*	*cdbe*	*dbec*	*ebad*
abec	*bacd*	*beca*	*cdea*	*dcab*	*ebca*
abed	*bace*	*becd*	*cdeb*	*dcae*	*ebcd*
acbd	*badc*	*beda*	*ceab*	*dcba*	*ebda*
acbe	*bade*	*bedc*	*cead*	*dcbe*	*ebdc*
acdb	*baec*	*cabd*	*ceba*	*dcea*	*ecab*
acde	*baed*	*cabe*	*cebd*	*dceb*	*ecad*
aceb	*bcad*	*cadb*	*ceda*	*deab*	*ecba*
aced	*bcae*	*cade*	*cedb*	*deac*	*ecbd*
adbc	*bcda*	*caeb*	*dabc*	*deba*	*ecda*
adbe	*bcde*	*caed*	*dabe*	*debc*	*ecdb*
adcb	*bcea*	*cbad*	*dacb*	*deca*	*edab*
adce	*bced*	*cbae*	*dace*	*decb*	*edac*
adeb	*bdac*	*cbda*	*daeb*	*eabc*	*edba*
adec	*bdae*	*cbde*	*daec*	*eabd*	*edbc*
aebc	*bdca*	*cbea*	*dbac*	*eacb*	*edca*
aebd	*bdce*	*cbed*	*dbae*	*eacd*	*edcb*

The number of Variations without Repetitions for n things is:—

For the 1st Class $= n$

— — 2nd Class $= n\,(n-1)$

— — 3rd Class $= n\,(n-1)\,(n-2)$

— — 4th Class $= n\,(n-1)\,(n-2)\,(n-3)$

.....

— — mth Class $= n\,(n-1)\,(n-2).....(n-m+1)$

X. THE BINOMIAL AND POLYNOMIAL THEOREMS FOR POSITIVE-INTEGER EXPONENTS.

(1) THE BINOMIAL THEOREM.

The Formula.

$$\text{I.}\quad (a \pm b)^n = a^n \pm \frac{n}{1} a^{n-1}b^1 + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2$$
$$\pm \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} a^{n-4}b^4$$
$$\pm \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} a^{n-5}b^5 + \ldots\ldots\ldots\ldots$$
$$\pm \frac{n(n-1)(n-2)\cdots\cdots 1}{1 \cdot 2 \cdot 3 \cdots\cdots n} b^n$$

As to $\pm$ which occur here, it is to be observed, that $+$ belongs to $(a+b)^n$, and $-$ to $(a-b)^n$. The last term is always b^n, and takes the sign $+$ or $-$ according as n is even or odd.

II. The number of terms of the series is $(n+1)$. The law of the Exponents of a and b is evident. The coefficients increase as far as the middle of the series, and then decrease in the same order and dimensions; so that the coefficients of the terms equally distant from the first and last, are equal. When n is even there is a middle term, and this term for $(a+b)^n$, is

$$+\frac{n(n-1)(n-2)\ldots\ldots\left(\frac{n}{2}+1\right)}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots \frac{n}{2}} a^{\frac{n}{2}} b^{\frac{n}{2}}$$

For $(a-b)^n$ this middle term is negative, when n is of the form $4m+2$, otherwise it is positive. When n is an odd number there are two middle terms, and these in $(a+b)^n$ are

$$+\frac{n(n-1)\cdots\frac{n+3}{2}}{1\cdot 2\quad\cdots\frac{n-1}{2}}a^{\frac{n+1}{2}}b^{\frac{n-1}{2}}+\frac{n(n-1)\cdots\frac{n+1}{2}}{1\cdot 2\quad\cdots\frac{n+1}{2}}a^{\frac{n-1}{2}}b^{\frac{n+1}{2}}$$

$$=\frac{n(n-1)\cdots\frac{n+3}{2}}{1\cdot 2\quad\cdots\frac{n-1}{2}}\left(+a^{\frac{n+1}{2}}b^{\frac{n-1}{2}}+a^{\frac{n-1}{2}}b^{\frac{n+1}{2}}\right)$$

For $(a-b)^n$, the first of the above terms is positive and the second negative, when n is of the form $4m+1$, and *vice versa*, when n is of the form $4m+3$.

III. The general $(m+1)$th term of the series is

$$\pm\frac{n(n-1)(n-2)\cdots\cdots(n-m+1)}{1\cdot 2\cdot 3\cdots\cdots\cdots\cdots m}a^{n-m}b^m$$

Of the signs $\pm$, the upper relates to all the terms of $(a+b)^n$, and in the odd terms also of $(a-b)^n$; but the lower to the even terms of $(a-b)^m$ only.

IV. The sum of the coefficients in the series for $(a+b)^n$, or $1+\frac{n}{1}+\frac{n(n-1)}{2}+\&c.=(1+1)^n=2^n$, and the sum of those for $(a-b)^n$, or $1-\frac{n}{1}+\frac{n(n-1)}{2}$ —&c. $=(1-1)^n=0$.

V. If $\frac{b}{a}$ be put $=Q$, and the first term of the series be denoted by A, the second by B, the third by C, and so on we shall have

See Emerson's Al. p. 139, also Barlow's Tables p. 26 where the index is $\frac{m}{n}$

$$(a \pm b)^n = A \pm \frac{n}{1} AQ + \frac{n-1}{2} BQ \pm \frac{n-2}{3} CQ + \frac{n-3}{4} DQ \pm \frac{n-4}{5} EQ \ldots\ldots$$

which formula is very convenient in practice, as we thoroughly deduce each term from the one immediately preceding.

EXAMPLES.

1. $(a \pm b)^1 = a \pm b$
2. $(a \pm b)^2 = a^2 \pm 2ab + b^2$
3. $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$
4. $(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$
5. $(a \pm b)^5 = a^5 \pm 5a^4b + 10a^3b^2 \pm 10a^2b^3 + 5ab^4 \pm b^5$
6. $(a \pm b)^6 = a^6 \pm 6a^5b + 15a^4b^2 \pm 20a^3b^3 + 15a^2b^4 \pm 6ab^5 + b^6$
7. $(a \pm b)^7 = a^7 \pm 7a^6b + 21a^5b^2 \pm 35a^4b^3 + 35a^3b^4 \pm 21a^2b^5 + 7ab^6 \pm b^7$
8. $(a \pm b)^8 = a^8 \pm 8a^7b + 28a^6b^2 \pm 56a^5b^3 + 70a^4b^4 \pm 56a^3b^5 + 28a^2b^6 \pm 8ab^7 + b^8$
9. $(a \pm b)^9 = a^9 \pm 9a^8b + 36a^7b^2 \pm 84a^6b^3 + 126a^5b^4 \pm 126a^4b^5 + 84a^3b^6 \pm 36a^2b^7 + 9ab^8 \pm b^9$
10. $(a \pm b)^{10} = a^{10} \pm 10a^9b + 45a^8b^2 \pm 120a^7b^3 + 210a^6b^4 \pm 252a^5b^5 + 210a^4b^6 \pm 120a^3b^7 + 45a^2b^8 \pm 10ab^9 + b^{10}$
11. $(1 \pm x)^{11} = 1 \pm 11x + 55x^2 \pm 165x^3 + 330x^4 \pm 462x^5 + 462x^6 \pm 330x^7 + 165x^8 \pm 55x^9 + 11x^{10} \pm x^{11}$
12. $(1 \pm x)^{12} = 1 \pm 12x + 66x^2 \pm 220x^3 + 495x^4 \pm 792x^5 + 924x^6 \pm 792x^7 + 495x^8 \pm 220x^9 + 66x^{10} \pm 12x^{11} + x^{12}$

13. $(5-4x)^4 = 625-2000x+2400x^2-1280x^3+256x^4$

14. $(3-2x^2)^6 = 729-2916x^2+4860x^4-4320x^6+2160x^8-576x^{10}+64x^{12}$

15. $(\frac{1}{2}x+2y)^7 = \frac{1}{128}x^7+\frac{7}{32}x^6y+\frac{21}{8}x^5y^2+\frac{35}{2}x^4y^3+70x^3y^4+168x^2y^5+224xy^6+128y^7$

16. $(a^3+3ab)^9 = a^{27}+27a^{25}b+324a^{23}b^2+2268a^{21}b^3+10206a^{19}b^4+30618a^{17}b^5+61236a^{15}b^6+78732a^{13}b^7+59049a^{11}b^8+19683a^9b^9$

17. $(3ac-2bd)^5 = 243a^5c^5-810a^4c^4bd+1080a^3c^3b^2d^2-720a^2c^2b^3d^3+240acb^4d^4-32b^5d^5$

18. $(5a^2c^2d-4abd^2)^4 = 625a^8c^8d^4-2000a^7bc^6d^5+2400a^6b^2c^4d^6-1280a^5b^3c^2d^7+256a^4b^4d^8$

19. $\left(\frac{2ac}{b^2}+\frac{1}{4}bc^2d\right)^6 = 64a^6c^6b^{-12}+48a^5c^7db^{-9}+15a^4c^8d^2b^{-6}+\frac{5}{2}a^3c^9d^3b^{-3}+\frac{15}{64}a^2c^{10}d^4+\frac{3}{256}ab^3c^{11}d^5+\frac{1}{4096}b^6c^{12}d^6$

20. $(\sqrt{a}\pm\sqrt{b})^4 = a^2+6ab+b^2\pm(4a+4b)\sqrt{ab}$

21. $(\sqrt{a}\pm\sqrt{b})^7 = (a^3+21a^2b+35ab^2+7b^3)\sqrt{a}\pm(7a^3+35a^2b+21ab^2+b^3)\sqrt{b}$

22. $(a+b)^n+(a-b)^n = 2\left(a^n+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^{n-4}b^4+\frac{n\ldots n-5}{1\ldots\ldots 6}a^{n-6}b^6+\ldots\ldots\right)$*

---

* The series 22, 23, 24, 25, 26, are continued till they terminate, or until the coefficients become 0.

23. $(a+b)^n-(a-b)^n=2\left(\frac{n}{1}a^{n-1}b+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3\right.$
$\left.+\frac{n\ldots n-4}{1\ldots\ldots 5}a^{n-5}b^5+\frac{n\ldots n-6}{1\ldots\ldots 7}a^{n-7}b^7+\ldots\ldots\right)$

24. $(a\pm b\sqrt{-1})^n=a^n-\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+$
$\frac{n\cdots n-3}{1\ldots\ldots 4}a^{n-4}b^4-$, &c. $\pm\left(\frac{n}{1}a^{n-1}b-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3\right.$
$\left.+\frac{n\ldots n-4}{1\ldots\ldots 5}a^{n-5}b^5-\text{ &c.}\right)\sqrt{-1}$

25. $(a+b\sqrt{-1})^n+(a-b\sqrt{-1})^n=$
$2\left[a^n-\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\frac{n\ldots\ldots n-3}{1\ldots\ldots 4}a^{n-4}b^4-\text{ &c.}\right]$*

26. $\frac{(a+b\sqrt{-1})^n-(a-b\sqrt{-1})^n}{\sqrt{-1}}=$
$2\left[na^{n-1}b-\frac{n\cdot n-1\cdot n-2}{1\cdot 2\cdot 3}a^{n-3}b^3+\frac{n\ldots\ldots n-4}{1\ldots\ldots 5}a^{n-5}b^5-\text{ &c.}\right]$

The Determination of Single Terms.

27. The 3rd term of $(a+b)^{15}$ is $= 105a^{13}b^2$
28. — 5th — — $(a+b)^{16}$ is $= 1820a^{12}b^4$
29. — 6th — — $(a-b)^{30}$ is $= -142506a^{25}b^5$
30. — 4th — — $(a-b)^{100}$ is $= -161700a^{97}b^3$
31. — 5th — — $(a^2-b^2)^{12}$ is $= 495a^{16}b^8$

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* Therefore $(a+b\sqrt{-1})^n+(a-b\sqrt{-1})^n$, also $[(a+b\sqrt{-1})^n-(a-b\sqrt{-1})^n]:\sqrt{-1}$ are real magnitudes.

32. The 9th term of $(2ab-cd)^{14}$ is $= 192192a^6b^6c^8d^8$

33. The middle term of $(a-b)^{16}$ is $= 12870a^8b^8$

34. — — — — $(a-b)^{18}$ is $= -48620a^9b^9$

35. The two middle terms of $(a-b)^{17}$ are
$$24310a^9b^8 - 24310a^8b^9$$

36. — — — — — — $(a-b)^{19}$ are
$$-92378a^{10}b^9 + 92378a^9b^{10}$$

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(2) THE POLYNOMIAL THEOREM.

THE FORMULA.

I. The nth power of the Polynomial $a+b+c+d+$ &c. $+x$ consists of the sum of all the *Combinations with Repetitions* of the magnitudes, a, b, c, $d \ldots . x$ for the nth Class, each taken with its number of Transpositions.

II. If, therefore, the Polynomial $a+b+c+d+\ldots.$ consist of m terms, then its power contains
$$\frac{m(m+1)\,(m+2)\ldots..(m+n-1)}{1\cdot 2\cdot 3\ldots\ldots\ldots\ldots n}$$
terms. The sum of the Exponents of a, b, c, $d \ldots .$ in each term is n, and the sum of the coefficients is m^n.

III. Putting for the sake of brevity $b+c+d+\ldots.x=p$. we have
$$(a+b+c+\ldots..+n)^n = (a+p)^n =$$
$$a^n + \frac{n}{1}a^{n-1}p + \frac{n(n-1)}{1\cdot 2}a^{n-2}p^2 + \ldots. + p^n$$
which Formula may be applied with advantage in many cases.

Examples.

1. $(a + b + c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2$

2. $(a + b + c)^3 = a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3$

3. $(a + b + c)^4 = a^4 + 4a^3b + 4a^3c + 6a^2b^2 + 12a^2bc + 6a^2c^2 + 4ab^3 + 12ab^2c + 12abc^2 + 4ac^3 + b^4 + 4b^3c + 6b^2c^2 + 4bc^3 + c^4$

4. $(a + b + c)^5 = a^5 + 5a^4b + 5a^4c + 10a^3b^2 + 20a^3bc + 10a^3c^2 + 10a^2b^3 + 30a^2b^2c + 30a^2bc^2 + 10a^2c^3 + 5ab^4 + 20ab^3c + 30ab^2c^2 + 20abc^3 + 5ac^4 + b^5 + 5b^4c + 10b^3b^2 + 10b^2c^3 + 5bc^4 + c^5$

5. $(a + b + c)^6 = a^6 + 6a^5b + 6a^5c + 15a^4b^2 + 30a^4bc + 15a^4c^2 + 20a^3b^3 + 60a^3b^2c + 60a^3bc^2 + 20a^3c^3 + 15a^2b^4 + 60a^2b^3c + 90a^2b^2c^2 + 60a^2bc^3 + 15a^2c^4 + 6ab^5 + 30ab^4c + 60ab^3c^2 + 60ab^2c^3 + 30abc^4 + 6ac^5 + b^6 + 6b^5c + 15b^4c^2 + 20b^3c^3 + 15b^2c^4 + 6bc^5 + c^6$

6. $(a + b + c)^7 = a^7 + 7a^6b + 7a^6c + 21a^5b^2 + 42a^5bc + 21a^5c^2 + 35a^4b^3 + 105a^4b^2c + 105a^4bc^2 + 35a^4c^3 + 35a^3b^4 + 140a^3b^3c + 210a^3b^2c^2 + 140a^3bc^3 + 35a^3c^4 + 21a^2b^5 + 105a^2b^4c + 210a^2b^3c^2 + 210a^2b^2c^3 + 105a^2bc^4 + 21a^2c^5 + 7ab^6 + 42ab^5c + 105ab^4c^2 + 140ab^3c^3 + 105ab^2c^4 + 42abc^5 + 7ac^6 + b^7 + 7b^6c + 21b^5c^2 + 35b^4c^3 + 35b^3c^4 + 21b^2c^5 + 7bc^6 + c^7$

7. $(a + b + c + d)^2 = a^2 + 2ab + 2ac + 2ad + b^2 + 2bc + 2bd + c^2 + 2cd + d^2$

8. $(a+b+c+d)^3 = a^3 + 3a^2b + 3a^2c + 3a^2d + 3ab^2 + 6abc + 6abd + 3ac^2 + 6acd + 3ad^2 + b^3 + 3b^2c + 3b^2d + 3bc^2 + 6bcd + 3bd^2 + c^3 + 3c^2d + 3cd^2 + d^3$

9. $(a+b+c+d)^4 = a^4 + 4a^3b + 4a^3c + 4a^3d + 6a^2b^2 + 12a^2bc + 12a^2bd + 6a^2c^2 + 12a^2cd + 6a^2d^2 + 4ab^3 + 12ab^2c + 12ab^2d + 12abc^2 + 24abcd + 12abd^2 + 4ac^3 + 12ac^2d + 12acd^2 + 4ad^3 + b^4 + 4b^3c + 4b^3d + 6b^2c^2 + 12b^2cd + 6b^2d^2 + 4bc^3 + 12bc^2d + 12bcd^2 + 4bd^3 + c^4 + 4c^3d + 6c^2d^2 + 4cd^3 + d^4$

10. $(a+b+c+d)^5 = a^5 + 5a^4b + 5a^4c + 5a^4d + 10a^3b^2 + 20a^2bc + 20a^3bd + 10a^3c^2 + 20a^3cd + 13a^3d^4 + 10a^2b^3 + 30a^2b^2c + 30a^2b^2d + 30a^2bc^2 + 60a^2bcd + 30a^2bd^2 + 10a^2c^3 + 30a^2c^2d + 30a^2cd^2 + 10a^2d^3 + 5ab^4 + 20ab^3c + 20ab^3d + 30ab^2c^2 + 60ab^2cd + 30ab^2d^2 + 20abc^3 + 60abc^2d + 60abcd^2 + 20abd^3 + 5ac^4 + 20ac^3d + 30ac^2d^2 + 20acd^3 + 5ad^4 + b^5 + 5b^4c + 5b^4d + 10b^3c^2 + 20b^3cd + 10b^3d^2 + 10b^2c^3 + 30b^2c^2d + 30b^2cd^2 + 10b^2d^3 + 5bc^4 + 20bc^3d + 30bc^2d^2 + 20bcd^3 + 5bd^4 + c^5 + 5c^4d + 10c^3d^2 + 10c^2d^3 + 5cd^4 + d^5$

11. $(a+2b+c)^3 = a^3 + 6a^2b - 3a^2c + 12ab^2 - 12abc + 3ac^2 + 8b^3 - 12b^2c + 6bc^2 - c^3$

12. $\left(3a - 5b - \frac{2c}{3}\right)^4 = 81a^4 - 540a^3b - 72a^3c + 1350a^2b^2 + 360a^2bc + 24a^2c^2 - 1500ab^3 - 600ab^2c - 80abc^2 - \frac{32}{9}ac^3 + 625b^4 + \frac{1000}{3}b^3c + \frac{200}{3}b^2c^2 + \frac{160}{27}bc^3 + \frac{16}{81}c^4$

13. $(7a^2 - 3ab + 4b^2)^3 = 343a^6 - 441a^4b + 777a^4b^2 - 531a^3b^3 + 444a^2b^4 - 144ab^5 + 64b^6$

14. $\left(\frac{2b^3c}{3a^4} + 7a^2b - \frac{1}{2}\right)^5 = \frac{32}{243}a^{-20}b^{15}c^5 + \frac{560}{81}a^{-4}b^{13}c^4 - \frac{40}{81}a^{-16}b^{12}c^4 + \frac{5920}{27}a^{-8}b^{11}c^3 - \frac{560}{27}a^{-10}b^{10}c^3 + \frac{20}{27}a^{-12}b^9c^3 + \frac{13720}{9}a^{-2}b^9c^2 - \frac{980}{3}a^{-4}b^8c^2 + \frac{70}{3}a^{-6}b^7c^7 - \frac{5}{3}a^{-8}b^6c^2 + \frac{24010}{3}a^4b^7c - \frac{6860}{3}a^2b^6c + 245b^5c - \frac{343}{3}a^{-2}b^4c + \frac{5}{24}a^{-4}b^3c + 16807a^{10}b^5 - \frac{12005}{2}a^8b^4 + \frac{1715}{2}a^6b^8 - \frac{245}{4}a^4b^2 + \frac{35}{16}a^2b - \frac{1}{32}$

15. $\left(\frac{2ab}{c^m} - 5a^3c^{3m} - 3ab^2 + \frac{b}{2a}\right)^3 = 8a^3b^3c^{-3m} - 60a^5b^2c^m - 36a^3b^4c^{-2m} + 6ab^3c^{-2m} + 150a^7bc^{5m} + 180a^5b^4c^{2m} - 30a^3b^2c^{2m} + 54a^3b^5c^{-m} - 18ab^4c^{-m} + \frac{5}{2}a^{-1}b^8c^{-m} - 125a^9c^{9m} - 225a^7b^7c^{6m} + \frac{75}{2}a^5bc^{6m} - 135a^5b^4c^{3m} + 45a^3b^3c^{3m} - \frac{15}{4}ab^2c^{3m} - 27a^3b^6 + \frac{27}{2}ab^5 - \frac{9}{4}a^{-1}b^4 + \frac{1}{8}a^{-3}b^3$

XI. PROGRESSIONS.

(1) ARITHMETICAL PROGRESSION.

I. Let a denote the first term, t the last, n the number of terms, d the common difference, and s the sum of an Arithmetical Progression; then we have

1. $t = a + (n - 1)\,d$

2. $s = (a + t)\frac{n}{2} = [2a + (n - 1)\,d]\frac{n}{2}$

By these two Formulæ, the values of t and s may be found when a, d, and n are given.

Examples.

N.	Given Values.			Required Values.	
1	$a=1$,	$d=1$,	$n=14$	$t=14$,	$s=105$
2	$a=2$,	$d=3$,	$n=17$	$t=50$,	$s=442$
3	$a=7$,	$d=\frac{1}{4}$,	$n=16$	$t=10\frac{3}{4}$,	$s=142$
4	$a=2\frac{1}{2}$,	$d=\frac{1}{3}$,	$n=100$	$t=35\frac{1}{2}$,	$s=1900$
5	$a=\frac{3}{4}$,	$d=\frac{1}{8}$,	$n=26$	$t=3\frac{7}{8}$,	$s=60\frac{1}{8}$
6	$a=\frac{5}{7}$,	$d=1\frac{2}{3}$,	$n=13$	$t=20\frac{5}{7}$,	$s=139\frac{2}{7}$
7	$a=-7$,	$d=3$,	$n=8$	$t=14$,	$s=28$
8	$a=-6$,	$d=\frac{3}{4}$,	$n=30$	$t=15\frac{3}{4}$,	$s=146\frac{1}{4}$
9	$a=\frac{1}{2}$,	$d=-\frac{1}{8}$,	$n=20$	$t=-1\frac{7}{8}$,	$s=-13\frac{3}{4}$
10	$a=3\frac{1}{3}$,	$d=-2\frac{5}{6}$,	$n=15$	$t=-36\frac{1}{3}$,	$s=-247\frac{1}{2}$
11	$a=0$,	$d=\frac{1}{2}$,	$n=11$	$t=5$,	$s=27\frac{1}{2}$
12	$a=-10$,	$d=-2$,	$n=6$	$t=-20$,	$s=-90$
13	$a=-\frac{3}{4}$,	$d=-\frac{7}{8}$,	$n=25$	$t=-21\frac{3}{4}$,	$s=-281\frac{1}{4}$

II. When 3 of the 5 magnitudes a, d, n, t, s are given, the other two may be found by the following Table:

Table of Formulæ for Arithmetical Progression.

N.	Given.	Required	Formulæ.
1	a, d, n	t	$t = a+(n-1)\,d$
2	a, d, s		$t = -\frac{1}{2}d \pm \sqrt{[2ds + (a-\frac{1}{2}d)^2]}$
3	a, n, s		$t = \frac{2s}{n} - a$
4	d, n, s		$t = \frac{s}{n} + \frac{(n-1)\,d}{2}$
5	a, d, n	s	$s = \frac{1}{2}n\,[2a+(n-1)\,d]$
6	a, d, t		$s = \frac{a+t}{2} + \frac{(t+a)\,(t-a)}{2d}$
7	a, n, t		$s = \frac{1}{2}n\,(a+t)$
8	d, n, t		$s = \frac{1}{2}n\,[2t-(n-1)\,d]$
9	a, n, t	d	$d = \frac{t-a}{n-1}$
10	a, n, s		$d = \frac{2s-2an}{n\,(n-1)}$
11	a, t, s		$d = \frac{(t+a)\,(t-a)}{(2s-t-a)}$
12	n, t, s		$d = \frac{2nt-2s}{n\,(n-1)}$

N.	Given.	Required	Formulæ.
13	$a,\ d,\ t$	n	$n = 1 + \frac{t-a}{d}$
14	$a,\ d,\ s$	n	$n = \frac{d-2a}{2d} \pm \sqrt{\left[\frac{2s}{d} + \left(\frac{2a-d}{2d}\right)^2\right]}$
15	$a,\ t,\ s$	n	$n = \frac{2s}{a+t}$
16	$d,\ t,\ s$	n	$n = \frac{2t+d}{2d} \pm \sqrt{\left[\left(\frac{2t+d}{2d}\right)^2 - \frac{2s}{d}\right]}$
17	$d,\ n,\ t$	a	$a = t - (n-1)\,d$
18	$d,\ n,\ s$	a	$a = \frac{s}{n} - \frac{(n-1)\,d}{2}$
19	$d,\ t,\ s$	a	$a = \frac{1}{2}d \pm \sqrt{[(t + \frac{1}{2}d)^2 - 2ds]}$
20	$n,\ t,\ s$	a	$a = \frac{2s}{n} - t$

What is meant by an Arithmetical Progression of the 1st, 2nd, 3rd, &c. Order? And how are the successive Orders derived from the Series of the 1st Order, a, $a+d$, $a+2d$, $a+3d$, &c.

What are Figurate Numbers? And how are they derived from the Series 1, $1+d$, $1+2d$, &c.? As particular cases, what are Polygonal and Pyramidal Numbers?*

* The general terms in the following series may be very easily verified, merely by Subtraction, or conversely by Addition. Thus we

Series of Figurates of the First Order.

1, 2, 3, 4, 5, 6 ..... n
1, 3, 5, 7, 9, 11 ..... $2n-1$
1, 4, 7, 10, 13, 16 ..... $3n-2$
1, 5, 9, 13, 17, 21 ..... $4n-3$
....
1, $1+d$, $1+2d$, $1+3d$ ..... $dn-d+1$

Polygonal Numbers.

1, 3, 6, 10, 15, 21 ..... $\frac{n(n+1)}{1 \cdot 2}$
1, 4, 9, 16, 25, 36 ..... n^2
1, 5, 12, 22, 35, 51 ..... $\frac{n(3n-1)}{1 \cdot 2}$
1, 6, 15, 28, 45, 66 ..... $n(2n-1)$
.....
1, $2+d$, $3+3d$, $4+6d$ ..... $\frac{n(dn-d+2)}{1 \cdot 2}$

Pyramidal Numbers.

1, 4, 10, 20, 35, 56 ..... $\frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$
1, 5, 14, 30, 55, 91 ..... $\frac{n(n+1)(2n+1)}{1 \cdot 2 \cdot 3}$

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need but subtract from the general term of each series that immediately preceding, and we shall obtain the general term of the series, from which the former is derived. For example, from the general term of Triangular Numbers $\frac{n \cdot (n+1)}{1 \cdot 2}$, we obtain the general term of the Natural Numbers $\frac{n \cdot (n+1)}{2} - \frac{(n-1) \cdot n}{2} = n$.

1, 6, 18, 40, 75, 126 ..... $\frac{n^2(n+1)}{1 \cdot 2}$

1, 7, 22, 50, 95, 161 ..... $\frac{n(n+1)(4n-1)}{1 \cdot 2 \cdot 3}$

.....

1, $3+d$, $6+4d$, $10+10d$ ..... $\frac{n(n+1)(dn-d+3)}{1 \cdot 2 \cdot 3}$

SERIES WHICH ARISE FROM THE CONTINUAL ADDITION OF THE NATURAL NUMBERS.

1, 2, 3, 4, 5, 6 ..... n

1, 3, 6, 10, 15, 21 ..... $\frac{n(n+1)}{1 \cdot 2}$

1, 4, 10, 20, 35, 56 ..... $\frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$

1, 5, 15, 35, 70, 126 ..... $\frac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4}$

&c. &c.

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(2) GEOMETRICAL PROGRESSION.

I. If a be the first term, e the common ratio, t the last term, n the number of terms, and s the sum of them, then

1. $t = ae^{n-1}$

2. $s = \frac{et-a}{e-1} = \frac{a(e^n-1)}{e-1}$

By means of these formulæ, t and s are found, when a, e, n are given.

N.	Given Values.			Required Values.	
1	$a=1,$	$e=2,$	$n=7$	$t=64,$	$s=127$
2	$a=4,$	$e=3,$	$n=10$	$t=78732,$	$s=118096$
3	$a=5,$	$e=4,$	$n=9$	$t=327680,$	$s=436905$
4	$a=9,$	$e=\frac{7}{4},$	$n=7$	$t=258\frac{2073}{4096},$	$s=591\frac{741}{4096}$
5	$a=6\frac{1}{4},$	$e=\frac{3}{2},$	$n=8$	$t=106\frac{403}{512},$	$s=307\frac{441}{512}$
6	$a=6,$	$e=\frac{3}{4},$	$n=6$	$t=1\frac{217}{512},$	$s=19\frac{373}{512}$
7	$a=8,$	$e=\frac{1}{2},$	$n=15$	$t=\frac{1}{2048},$	$s=15\frac{2047}{2048}$
8	$a=3\frac{1}{2},$	$e=\frac{3}{5},$	$n=8$	$t=\frac{15309}{131250},$	$s=8\frac{45257}{75625}$
9	$a=\frac{5}{6},$	$e=\frac{2}{3},$	$n=11$	$t=\frac{2560}{177147},$	$s=2\frac{1666927}{354294}$
10	$a=3,$	$e=\frac{7}{5},$	$n=25$	$t=9642{\cdot}59\ldots,$	$s=33741{\cdot}59\ldots$
11	$a=7\frac{1}{2},$	$e=\frac{27}{20},$	$n=31$	$t=60964{\cdot}11\ldots,$	$s=235125{\cdot}85\ldots$
12	$a=63,$	$e=\frac{157}{132},$	$n=58$	$t=1238530{\cdot}19\ldots,$	$s=7777637{\cdot}01\ldots$
13	$a=5560,$	$e=\frac{9}{11},$	$n=40$	$t=2{\cdot}219309\ldots,$	$s=30570{\cdot}01310\ldots$
14	$a=393\frac{1}{3},$	$e=\frac{5}{12},$	$n=17$	$t=0{\cdot}0003246241\ldots,$	$s=674{\cdot}2854824\ldots$
15	$a=1,$	$e=\frac{1}{2},$	$n=\infty$	$t=0,$	$s=2$
16	$a=40,$	$e=\frac{3}{7},$	$n=\infty$	$t=0,$	$s=70$
17	$a=9,$	$e=\frac{2}{3},$	$n=\infty$	$t=0,$	$s=27$

In the expressions 10, 11, 12, 13, 14, the values of t are most easily found by Logarithms; whence those of s are soon obtained.

What is the sum of n terms of the Geometrical Progression $a, b, \frac{b^2}{a}, \frac{b^3}{a^2}, \ldots \frac{b^{n-1}}{a^{n-2}}$? And what is the sum when the number of terms is infinite.

Ans. The sum of the finite Progression is $\frac{b^n - a^n}{(b-a)a^{n-2}} = \frac{a^n - b^n}{(a-b)a^{n-2}}$; of the infinite one $= \frac{a^2}{a-b}$.

What is the sum of the Infinite Progression

$$a - b + \frac{b^2}{a} - \frac{b^3}{a^2} + \frac{b^4}{a^3} - \text{\&c.} \qquad \textit{Ans.} \ \frac{a^2}{a+b}.$$

What is the Vulgar Fraction which is equivalent to the recurring decimal one 0·868686, &c. = 86 (0·01 + 0·0001 + 0·000001 + ...)? *Ans.* $\frac{86}{99}$.

And which to 0·375375375 .....?

Ans. $\frac{375}{999} = \frac{125}{333}$.

And again, which to 0·142857 ..... whose period is 142857? *Ans.* $\frac{142857}{999999} = \frac{1}{7}$.

Every recurring Decimal Fraction may be transformed to a Vulgar Fraction.

II. When any three of the magnitudes a, e, n, t, s are given, the other two may be found, as in the following Table:

Table of Forms for Geometrical Progression.

N.	Given.	Required	Formulæ.
1	$a,\ e,\ n$	t	$t = ae^{n-1}$
2	$a,\ e,\ s$	t	$t = \dfrac{a + (e-1)s}{e}$
3	$a,\ n,\ s$	t	$t\,(s-t)^{n-1} - a(s-a)^{n-1} = 0$
4	$e,\ n,\ s$	t	$t = \dfrac{(e-1)se^{n-1}}{e^n - 1}$
5	$a,\ e,\ n$	s	$= \dfrac{a(e^n - 1)}{e-1}$
6	$a,\ e,\ t$	s	$s = \dfrac{et - a}{e-1}$
7	$a,\ n,\ t$	s	$s = \dfrac{\sqrt[n]{t^{n-1}} - \sqrt[n]{a^{n-1}}}{\dfrac{1}{t^{n-1}} - \dfrac{1}{a^{n-1}}}$
8	$e,\ n,\ t$	s	$s = \dfrac{t(e^n - 1)}{(e-1)e^{n-1}}$
9	$e,\ n,$	a	$a = \dfrac{t}{e^{n-1}}$
10	$e,\ n,\ s$	a	$a = \dfrac{(e-1)s}{e^n - 1}$
11	$e,\ t,\ s$	a	$a = et - (e-1)s$
12	$n,\ t,\ s$	a	$a(s-a)^{n-1} - t(s-t)^{n-1} = 0$

N.	Given.	Required	Formulæ.
13	a, n, t	e	$e = \sqrt[n-1]{\frac{t}{a}}$
14	a, n, s	e	$e^n - \frac{s}{a}e + \frac{s-a}{a} = 0$
15	a, t, s	e	$e = \frac{s-a}{s-t}$
16	n, t, s	e	$e^n - \frac{s}{s-t}e^{n-1} + \frac{t}{s-t} = 0$
17	a, e, t	n	$n = \frac{\log t - \log a}{\log e} + 1$
18	a, e, s	n	$n = \frac{\log [a + (e-1)s] - \log a}{\log e}$
19	a, t, s	n	$n = \frac{\log t - \log a}{\log (s-a) - \log (s-t)} + 1$
20	e, t, s	n	$n = \frac{\log t - \log [et - (e-1)s]}{\log e} + 1$

XII. CONTINUED FRACTIONS.

(1) CONTINUED FRACTIONS IN GENERAL.

I. A Continued Fraction is of the following form:—

$$\cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \text{\&c.}}}}}}$$

the meaning of which is perceptible from the manner in which it is written. It is also assumed that a, b, c, d, e, &c. which are usually called Quotients, are all positive Integers. The fractions $\frac{1}{a}$, $\frac{1}{a+\frac{1}{b}}$, $\frac{1}{a+\frac{1}{b+\frac{1}{c}}}$, &c. are Approximating Fractions, and the farther they are continued the more nearly do they approach the given one.

II. If these Approximating Fractions be transformed into Vulgar Fractions, we shall have the following approximations to the given fractions :—

1. $\frac{1}{a}$
2. $\frac{b}{ab+1}$
3. $\frac{bc+1}{(ab+1)c+a}$
4. $\frac{(bc+1)d+b}{(abc+c+a)d+ab+1}$
5. $\frac{(bcd+d+b)e+bc+1}{(abcd+cd+ad+ab+1)e+abc+c+a}$

&c. &c.

III. These values may be derived one from another. Let $\frac{R}{S}$ be the $(n-1)$th of them, and $\frac{T}{V}$ the $(n-2)$th; farther let q be the nth of the quotients a, b, c, d, &c.; then the nth Approximating Fraction will be $= \frac{qR+T}{qS+V}$.

IV. An Approximating Fraction thus found is always in its simplest form, its factors having no common measure·

V. The Approximating Values are alternately greater and less than the true one.

VI. The difference between two successive Approximating Values is alternately positive and negative; it is always a fraction whose numerator is 1, and whose denominator is the product of the two values.

VII. If to get the Approximating Value we employ all the quotients a, b, c, &c. then we shall have obtained the true value of the Continued Fraction.

VII. To transform any magnitude whatever X, into a Continued Fraction, we give it the form $a + \frac{1}{x}$, where a is the greatest Integer contained in X, and may be 0, as in the case where X is < 1. Again for x we put $= a' + \frac{1}{x'}$, $x' = a'' + \frac{1}{x''}$, &c. a', a'', &c. being the greatest Integers contained in x, x', &c. respectively; thus obtaining

$$X = a + \cfrac{1}{a' + \cfrac{1}{a'' + \cfrac{1}{a''' + \text{\&c.}}}}$$

If X admit of being represented by a Vulgar Fraction, the continued Fraction will terminate; if not it will proceed to infinity.

IX. If q be the denominator of any Approximating Value, the difference between X and this value is always

less than $\frac{1}{q^2}$. Hence we have a sure test whereby to ascertain the degree of the Approximation.

TRANSFORMATION OF VULGAR INTO CONTINUED FRACTIONS.

The following Table gives a clear idea of the application of the general principle, as explained in VIII.

$$\frac{351}{965} = \frac{1}{2+\frac{263}{351}}, \quad \frac{263}{351} = \frac{1}{1+\frac{88}{263}}, \quad \frac{88}{263} = \frac{1}{2+\frac{87}{88}}, \quad \frac{87}{88} = \frac{1}{1+\frac{1}{87}}$$

Therefore
$$\frac{351}{965} = \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{87}}}}}$$

EXAMPLES.

N.	Given Fractions.	Quotients.	Approximating Fractions.
1	$\frac{351}{965}$	2, 1, 2, 1, 87	$\frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{4}{11}$
2	$\frac{251}{764}$	3, 22, 1, 4, 2	$\frac{1}{3}, \frac{22}{67}, \frac{53}{70}, \frac{114}{347}$
3	$\frac{1769}{5537}$	3, 7, 1, 2, 4, 5, 1, 2	$\frac{1}{3}, \frac{7}{22}, \frac{8}{25}, \frac{23}{72}, \frac{100}{313}, \frac{523}{1637}, \frac{623}{1950}$

N.	Given Fractions.	Quotients.	Approximating Fractions.
4	$\frac{907}{18564}$	20, 2, 7, 5, 2, 1, 3	$\frac{1}{20}, \frac{2}{41}, \frac{15}{307}, \frac{77}{1576}, \frac{169}{3459}, \frac{246}{5035}$
5	$\frac{1947}{3359}$	1, 1, 2, 1, 1, 1, 3, 2, 1, 1, 2, 3	$\frac{1}{1}, \frac{1}{2}, \frac{3}{5}, \frac{4}{7}, \frac{7}{12}, \frac{11}{19}, \frac{40}{69}, \frac{91}{157}, \frac{131}{226}, \frac{222}{383}, \frac{575}{992}$
6	$\frac{587}{1943}$	3, 3, 4, 2, 3, 1, 1, 2	$\frac{1}{3}, \frac{5}{10}, \frac{13}{43}, \frac{29}{96}, \frac{100}{331}, \frac{129}{427}, \frac{229}{758}$
7	$\frac{5065}{13891}$	2, 1, 2, 1, 7, 1, 1, 1, 2, 1, 13	$\frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{4}{11}, \frac{31}{85}, \frac{35}{96}, \frac{66}{181}, \frac{101}{277}, \frac{268}{735}, \frac{369}{1012}$
8	$\frac{5743}{80937}$	14, 10, 1, 2, 1, 3, 3, 3, 3	$\frac{1}{14}, \frac{10}{141}, \frac{11}{155}, \frac{32}{451}, \frac{43}{606}, \frac{161}{2269}, \frac{526}{7415}, \frac{1739}{24508}$
9	$\frac{13957}{59476}$	4, 3, 1, 4, 1, 2, 1, 11, 2, 6	$\frac{1}{4}, \frac{3}{13}, \frac{4}{17}, \frac{19}{81}, \frac{23}{98}, \frac{65}{277}, \frac{88}{375}, \frac{1033}{4402}, \frac{2154}{9179}$
10	$\frac{3215763}{94218374}$	29, 3, 2, 1, 8, 1, 1, 6, &c.	$\frac{1}{29}, \frac{3}{88}, \frac{7}{205}, \frac{10}{293}, \frac{87}{2549}, \frac{97}{2842}, \frac{184}{6391}$, &c.

The Lunar Month, or the time in which the Moon actually completes its revolution in the starry heaven, calculated on the average of 100 years, contains 27·321661 days. Therefore, in 27.321661 days it would perform 1000000 revolutions. How will this relation, expressed by such large numbers, admit of being denoted by less?

Ans. The Approximating Fractions of 27·321661 are $\frac{27}{1}, \frac{82}{3}, \frac{765}{28}, \frac{3907}{143}$, &c.; the third of which shows that the moon revolves 28 times in 765 days, which only differs from the truth by something above 0·0001 days.

According to Laplace, the greatest Mathematician living, the sidereal revolution of Mercury is 87·969255, and that of Venus 224·700817 days, how will these periods admit of being represented by less numbers?

Ans. That of Mercury by $\frac{87}{1}, \frac{88}{1}, \frac{2815}{32}$, &c.; that of Venus by $\frac{224}{1}, \frac{225}{1}, \frac{674}{3}, \frac{1573}{7}, \frac{2247}{10}, \frac{26290}{117}$, &c. The fractions $\frac{2815}{32}$ and $\frac{26290}{117}$ give them sufficiently exact.

The circumference of a circle is to its diameter as 3·1415926535 to 1. How is this proposition to be expressed in smaller numbers?

Ans. By 3 : 1; 22 : 7; 333 : 106; 355 : 113; 103993 : 33102 and so on.

(3) TRANSFORMATION OF $\sqrt{A}$ INTO A CONTINUED FRACTION.

It is assumed that A is an Integer. The following Scheme explains the application of the Principle in VIII to this case.

$$X = \sqrt{19} = 4 + \frac{\sqrt{19}-4}{1}\left(=\frac{1}{x}\right)$$

$$x = \frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{3} = 2 + \frac{\sqrt{19}-2}{3}\left(=\frac{1}{x^{I}}\right)$$

$$x^{I} = \frac{3}{\sqrt{19}-2} = \frac{\sqrt{19}+2}{5} = 1 + \frac{\sqrt{19}-3}{5}\left(=\frac{1}{x^{II}}\right)$$

$$x^{II} = \frac{5}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{2} = 3 + \frac{\sqrt{19}-3}{2}\left(=\frac{1}{x^{III}}\right)$$

$$x^{III} = \frac{2}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{5} = 1 + \frac{\sqrt{19}-2}{5}\left(=\frac{1}{x^{IV}}\right)$$

$$x^{IV} = \frac{5}{\sqrt{19}-2} = \frac{\sqrt{19}+2}{3} = 2 + \frac{\sqrt{19}-4}{3}\left(=\frac{1}{x^{V}}\right)$$

$$x^{V} = \frac{3}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{1} = 8 + \frac{\sqrt{19}-4}{1}\left(=\frac{1}{x^{VI}}\right)$$

$$x^{VI} = \frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{3} = 2 + \&c.$$

The Quotients here are 4, 2, 1, 3, 1, 2, 8, 2, &c.

Examples.

N.	Given Roots.	Quotients.	Approximating Fractions.
1	$\sqrt{28}$	5, 3, 2, 3, 10, &c.	$\frac{5}{1}, \frac{16}{3}, \frac{37}{7}, \frac{127}{24}, \frac{1307}{247}$ &c.
2	$\sqrt{31}$	5, 1, 1, 3, 5, 3, 1, 1, 10, &c.	$\frac{5}{1}, \frac{6}{1}, \frac{11}{2}, \frac{39}{7}, \frac{206}{37}, \frac{657}{118}, \frac{863}{155}, \frac{1520}{273}, \frac{16063}{2885}$, &c.

N.	Given Roots.	Quotients.	Approximating Fractions.
3	$\sqrt{44}$	6, 1, 1, 1, 2, 1, 1, 1, 12, &c.	$\frac{6}{1}, \frac{7}{1}, \frac{13}{2}, \frac{20}{3}, \frac{53}{8}, \frac{73}{11}, \frac{126}{19}, \frac{199}{30}, \frac{2514}{379}$, &c.
4	$\sqrt{45}$	6, 1, 2, 2, 2, 1, 12, &c.	$\frac{6}{1}, \frac{7}{1}, \frac{20}{3}, \frac{47}{7}, \frac{114}{17}, \frac{161}{24}, \frac{2046}{305}$, &c.
5	$\sqrt{52}$	7, 4, 1, 2, 1, 4, 14, &c.	$\frac{7}{1}, \frac{29}{4}, \frac{36}{5}, \frac{101}{14}, \frac{137}{19}, \frac{649}{90}, \frac{9223}{1279}$, &c.
6	$\sqrt{53}$	7, 3, 1, 1, 3, 14, &c.	$\frac{7}{1}, \frac{22}{3}, \frac{29}{4}, \frac{51}{7}, \frac{182}{21}, \frac{2599}{357}$, &c.
7	$\sqrt{59}$	7, 1, 2, 7, 2, 1, 14, &c.	$\frac{7}{1}, \frac{8}{1}, \frac{23}{3}, \frac{169}{22}, \frac{361}{47}, \frac{530}{69}, \frac{7781}{1013}$, &c.
8	$\sqrt{67}$	8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16, &c.	$\frac{8}{1}, \frac{41}{5}, \frac{90}{11}, \frac{131}{16}, \frac{221}{27}, \frac{1678}{205}, \frac{1899}{232}, \frac{3577}{437}, \frac{9053}{1106}, \frac{48842}{5967}, \frac{790525}{96578}$, &c.

By an attentive examination of the Quotients of $\sqrt{A}$, the following properties will be perceived:

1. The Quotients recur in Periods, and it will not be necessary to go beyond the first Period. The Period begins with the second Quotient and ends with that which is twice as great as the first.

2. Omitting the last Quotient of the Period, the others have the following order:

$$a, \beta, \gamma, \delta, \varepsilon \ldots\ldots\ldots \varepsilon, \delta, \gamma, \beta, a.$$

3. In general, let $\frac{p}{q}$ denote an Approximating Fraction belonging to the last quotient but one of any period; then we have

$$p^2 - Aq^2 = \pm 1.$$

The above examples afford sufficient illustration; for instance, $127^2 - 28 \times 24^2 = +1$, $1520^2 - 31 \times 273^2 = +1$, $199^2 - 44 \times 30^2 = +1$, $161^2 - 45 \times 24^2 = +1$, $649^2 - 52 \times 90^2 = +1$, $182^2 - 53 \times 25^2 = -1$, $530^2 - 59 \times 69^2 = +1$, $48842^2 - 67 \times 5967^2 = +1$.

4. When the Period consists of an even number of Quotients, $p^2 - Aq^2 = +1$ for all the Periods. But if that number be odd, then $p^2 - Aq^2 = -1$ and $+1$ alternately for such Periods.

5. In determining the Quotients, we always have Integers to calculate with and not Fractions.

Continued Fractions have indeed many remarkable properties, and are susceptible of very important applications. By them, for instance, as has been already seen, we are enabled to reduce the Factors of Fractions, be they ever so large, to smaller approximate ones of any required degree of accuracy. Moreover, the Approximating Fractions obtained by this method, of any given Fraction, are, as it may be strictly proved, expressed in smaller numbers, for the same degree of accuracy, than the Approximations found by any other known method.

SECTION II.

CONTAINING

ALGEBRAICAL EQUATIONS.

XII. THE ACCURATE SOLUTION OF EQUATIONS, WITH SOME PRELIMINARY OBSERVATIONS.

(1) EQUATIONS IN GENERAL.

I. AN Equation considered generally, is merely an opposite arrangement of two expressions, and these expressions are called its Members. An Equation, unless its Members are *identical*, is either *Analytical* or *Algebraical*.

II. An *Analytical Equation* is one, in which the equality of its members can be *verified* only, and solely, by evolution or explication of its letters and abstract characters, or by a series of characters. Of this kind of Equations are almost all those which occur in the preceding Section.

III. An *Algebraical Equation*, on the contrary, is such as can only be verified by substituting for the letters

therein, certain determinate numerical values, and not by any taken at pleasure. Of this kind are those which occur in this Section.

IV. Moreover, *Algebra* is the science by which magnitudes sought are obtained from given magnitudes. Hence by it the Solution of Problems is effected. The Problem is first analyzed; relations between the required and given magnitudes are then sought; and these relations are expressed by means of letters and other symbols in the form of Equations. The last letters of the alphabet denote the magnitudes required; the first, those which are given.

V. *Modern Analysis* treats only of the evolution of forms, and therefore merely of Analytical Equations. It is also essentially different from *Algebra*, although in their higher branches they meet, as it were, and mutually assist each other.

VI. The Analysis of the *Ancients* is not essentially different from our Algebra; the latter having merely the advantage (and a very important one it is) of a scientific Notation. Both derive the thing sought from things given, by finding their mutual relations, and reducing them through various degrees of simplicity.

VII. The Solution of Equations is shown in most Elementary Treatises. It may be remarked, however, for the sake of beginners, that *To determine the magnitudes required from those given, there ought to be as many equations as things required; That of these equations none must*

be Analytical; and that none must be so constituted as to be dependent on or the consequence of the others. Equations not answering these conditions must be rejected as useless in the Solution of Problems.

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(1) SIMPLE EQUATIONS.

(a) With one Unknown Magnitude.

1. $ax \pm b = c$

   $x = \dfrac{c \mp b}{a}$

2. $3a + x - 5b + 2 = 7b - a + c + 6$

   $x = 12b - 4a + c + 4$

3. $7 - 9a - 5x + 3cd + x = \frac{7}{4} - 3a - 2cd - 2x$

   $x = \frac{21}{8} - 3a + \frac{5}{2}cd$

4. $8x - 5 = 13 - 7x$

   $x = 1\frac{1}{5}$

5. $13\frac{3}{4} - \dfrac{x}{2} = 2x - 8\frac{3}{4}$

   $x = 9$

6. $2x + 7 + \frac{3}{2}x = 6x - 23$

   $x = 12$

7. $12\frac{1}{4} + 3x - 6 - \dfrac{7x}{3} = \dfrac{3x}{4} - 5\frac{3}{8}$

   $x = 139\frac{1}{2}$

8. $-6\frac{1}{3}x + 158\frac{1}{2} - 10x = -\dfrac{37x}{6} + 19 + \frac{3}{5}x$

   $x = 13\frac{59}{233}$

9. $8\frac{3}{4} + \frac{3x}{7} - \frac{5}{6} + 2x - \frac{12x}{5} + 13 + \frac{x}{4} = 0$

$x = -75\frac{10}{117}$

10. $\frac{3x}{5} - \frac{7x}{10} + \frac{3x}{4} - \frac{7x}{8} = -15$

$x = 66\frac{2}{3}$

11. $3\frac{2}{3} - x - \frac{9x}{2} + 8 = -17 - \frac{3x}{5} + \frac{3}{2}x$

$x = 4\frac{23}{48}$

12. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 7x - 712 + \frac{x}{5}$

$x = 116\frac{148}{367}$

13. $11\frac{1}{2}x = \frac{11}{8}x + 66\frac{7}{8} - 5x - 9\frac{1}{4}$

$x = 3\frac{91}{121}$

14. $-\frac{16}{7}x = -\frac{1}{2}x + 412\frac{1}{2} - \frac{3}{5}x - 316\frac{1}{2}$

$x = -80\frac{80}{83}$

15. $32\frac{1}{10}x + 176\frac{3}{4} - x = 19\frac{1}{3}x + 7345 - \frac{2x}{3}$

$x = 576\frac{599}{746}$

16. $3{\cdot}25x - 5{\cdot}007 - x = 0{\cdot}2 - 0{\cdot}34x$

$x = 2{\cdot}010424....$

17. $13{\cdot}2x - \frac{3x}{4} + 7{\cdot}6953 = \frac{x}{5} + 7834{\cdot}5$

$x = 638{\cdot}92283$

18. $\frac{7{\cdot}53x}{18} - 100 = \frac{2x}{5} - 3{\cdot}86 - \frac{x}{6}$

$x = 519{\cdot}67567....$

19. $ax + c = bx + d$

$$x = \frac{d - c}{a - b}$$

20. $\frac{f^2 x}{cg} - \frac{a^2}{f} + cx = \frac{hx}{g} - c + (a + c)\,x.$

$$x = \frac{(a^2 - cf)\,cg}{(f^2 - ch - acg)\,f}$$

21. $x = a + \frac{bc}{d} + \frac{cfx}{de}$

$$x = \frac{(ad + bc)\,e}{de - cf}$$

22. $\frac{ex}{f} + \frac{cx}{d} + \frac{ax}{b} - g = h$

$$x = \frac{(h + g)\,bdf}{b(de + cf) + adf}$$

23. $\frac{5}{6}ab + \frac{4}{5}ac - \frac{2}{3}cx = \frac{3}{4}ac + 2ab - 6cx$

$$x = \frac{(70b - 3c)\,a}{320c}$$

24. $\frac{x}{a} - 1 - \frac{dx}{c} + 3ab = 0$

$$x = \frac{ac\,(1 - 3ab)}{c - ad}$$

25. $\frac{ace}{d} - \frac{(a + b)^2 x}{a} - bx = ae - 3bx$

$$x = \frac{a^2 e\,(c - d)}{(a^2 + b^2)\,d}$$

26. $\frac{a + 3x}{4a} - \frac{7a - 5x}{6b} + 3 - \frac{9x}{4} = \frac{x}{ab} + \frac{5x}{6b}$

$$x = \frac{39ab - 14a^2}{27ab - 9b + 12}$$

27. $5a^3cx + ac^2x - 5abc^2 + 3abc^2 - 3a^3c^3 = 5a^2bcx + bc^2x - 3a^2bc^3 - 5a^2c^2$

$$x = \frac{3a^2c^2 - 5ac}{5a^2 + c}$$

28. $2a^2b^2c + ab^2x - 2ab^3c - abc^2d - 3a^3x = (b^3 - 3a^2b)x - b^2c^2d$

$$x = \frac{2ab^2c - bc^2d}{3a^2 - b^2}$$

29. $\frac{ab}{x} = bc + d + \frac{1}{x}$

$$x = \frac{ab - 1}{bc + d}$$

30. $\frac{3a+x}{x} - 5 = \frac{6}{x}$

$$x = \frac{3a - 6}{4}$$

31. $\frac{a^2x}{b - c} + dc = bx - ac$

$$x = \frac{c(a + d)(b - c)}{b(b - c) - a^2}$$

32. $c = a + \frac{m(a - x)}{3a + x}$

$$x = \frac{a(m - 3c + 3a)}{c - a + m}$$

33. $\frac{a(d^2 + x^2)}{dx} = ac + \frac{ax}{d}$

$$x = \frac{d}{c}$$

34. $\dfrac{cx^n}{a+bx} = \dfrac{fx^n}{d+ex}$

$$x = \frac{cd - af}{bf - ce} = \frac{af - cd}{ce - bf}$$

35. $\dfrac{7x^n}{x-1} = \dfrac{6x^{n+1} + x^n}{x+1} - \dfrac{3x^n + 6x^{n+2}}{x^2 - 1}$

$$x = -\tfrac{11}{12}$$

36. $\dfrac{3a - 5x}{a - c} + \dfrac{2a - x}{d} = \dfrac{a + f}{a - c} - dx$

$$x = \frac{d(f - 2a) - 2a(a - c)}{(a - c)(d^2 - 1) - 5d}$$

37. $\dfrac{a}{bx} + \dfrac{c}{dx} + \dfrac{e}{fx} + \dfrac{g}{hx} = k$

$$x = \frac{adfh + bcfh + bdeh + bdfg}{bdfhk}$$

38. $\dfrac{3abc}{a+b} + \dfrac{a^2b^2}{(a+b)^3} + \dfrac{(2a+b)b^2x}{a(a+b)^2} = 3cx + \dfrac{bx}{a}$

$$x = \frac{ab}{a+b}$$

39. $\dfrac{bx}{2b - a} - \dfrac{(3bc + ad)x}{2ab(a+b)} - \dfrac{5ab}{3c - d} = \dfrac{(3bc - ad)x}{2ab(a-b)}$

$$- \frac{5a(2b - a)}{a^2 - b^2}$$

$$x = \frac{5a(2b - a)}{3c - d}$$

40. $(a + x)(b + x) - a(b + c) = \dfrac{a^2c}{b} + x^2$

$$x = \frac{ac}{b}$$

41. $\sqrt[m]{x} = a$

$$x = a^m$$

42. $\sqrt[m]{(ax + b)} = \sqrt[m]{(cx + d)}$

$$x = \frac{d - b}{a - c}$$

43. $h\sqrt[3]{(ax - b)} = k\sqrt[3]{(cx + dx - f)}$

$$x = \frac{bh^3 - fk^3}{ah^3 - (c + d)k^3}$$

44. $\sqrt[3]{(a^2 + c)} = \sqrt[4]{\frac{a^2+c}{d(x + g)}}$

$$x = \frac{1}{d\sqrt[3]{(a^2+c)}} - g$$

45. $\sqrt[m]{(a + x)} = \sqrt[2m]{(x^2 + 5ax + b^2)}$

$$x = \frac{a^2 - b^2}{3a}$$

46. $c + b\sqrt[m]{(x + d)} = f$

$$x = \left(\frac{f - c}{b}\right)^m - d$$

47. $\frac{ax}{b}\sqrt{(f^2x^2 + d^2)} + \frac{afx^2}{b} = cx$

$$x = \frac{b^2c^2 - a^2d^2}{2abcf} = \frac{(bc + ad)(bc - ad)}{2abcf}$$

Logarithmic Equations.

48. $a^x = b$

$$x = \frac{\log b}{\log a}$$

49. $a^{mx}b^{nx} = c$

$$x = \frac{\log c}{m \log a + n \log b} = \frac{\log c}{\log a^m b^n}$$

50. $a^{mx+f}\, b^{nx+g} = c^{px+h}\, d^{qx+k}$

$$x = \frac{h \log c + k \log d - f \log a - g \log b}{m \log a + n \log b - p \log c - q \log d} =$$

$$\left(\log \frac{c^h d^k}{a^f b^g}\right) : \left(\log \frac{a^m b^n}{c^p d^q}\right)$$

51. $3^x = 177147$

$$x = 11$$

52. $2^x = 769$

$$x = 9{\cdot}586839 \ldots .$$

53. $\left(\frac{3}{4}\right)^x = 51\frac{1}{2}$

$$x = -\ 13{\cdot}701172 \ldots .$$

54. $\left(\frac{756}{345}\right)^{\frac{3x}{2}} = 54783$

$$x = 9{\cdot}272299$$

55. $\left(\frac{21}{20}\right)^x \left(\frac{3}{5}\right)^{\frac{7x}{2}} = \frac{7}{12}$

$$x = 0{\cdot}309928 \ldots .$$

56. $\left(\frac{295}{867}\right)^{3-x} = 632 \cdot \left(\frac{56}{39}\right)^{\frac{3x}{9}}$

$$x = 11{\cdot}040270 \ldots .$$

57. $3^{2x} \cdot 5^{6x-7} = 9^{x-2} \cdot y^{1-x}$

$$x = 0{\cdot}759965 \cdots$$

(*b*) *With more Unknown Magnitudes.*

1. $\left\{\begin{matrix} x + y = a \\ x - y = b \end{matrix}\right\}$

$$x = \frac{a+b}{2},\ y = \frac{a-b}{2}$$

2. $\left\{\begin{matrix} 3x + 2y = 118 \\ x + 5y = 191 \end{matrix}\right\}$

$$x = 16,\ y = 35$$

3. $\left\{\begin{matrix} 7x + \frac{5}{2}y = 411\frac{1}{2} \\ 39x - 14y = -935\frac{9}{10} \end{matrix}\right\}$

$$x = 17\tfrac{1}{2},\ y = 115\tfrac{3}{5}$$

4. $\left\{\begin{matrix} 5x - 8\frac{1}{2} = 7y - 44 \\ 2x = y + \frac{5}{7} \end{matrix}\right\}$

$$x = 4\tfrac{1}{2},\ y = 8\tfrac{2}{7}$$

5. $\left\{\begin{matrix} 5\frac{3}{4}y - 11x = 4y + 117\frac{1}{8} \\ 8x + 175 = 2y \end{matrix}\right\}$

$$x = 9,\ y = 123\tfrac{1}{2}$$

6. $\left\{\begin{matrix} 7y = 2x - 3y \\ 19x = 60y + 621\frac{1}{4} \end{matrix}\right\}$

$$x = 88\tfrac{3}{4},\ y = 17\tfrac{3}{4}$$

7. $\left\{\begin{matrix} 13x + 7y - 341 = 7\frac{1}{2}y + 43\frac{1}{2}x \\ 2x + \frac{1}{2}y = 1 \end{matrix}\right\}$

$$x = -12,\ y = 50$$

8. $\left\{\begin{matrix} 113\frac{1}{2}x - 27\frac{5}{7}y = 10y + 5488\frac{4}{7} \\ 9y - 347 = 5x - 420 \end{matrix}\right\}$

$$x = 56,\ y = 23$$

9. $\left\{\begin{array}{l} 168\frac{15}{14} - 19x + \frac{5}{11}y = 12\frac{3}{4}x + 1084 \\ \frac{1}{3}x - 149\frac{1}{2} = 319\frac{2}{9} - \frac{7}{2}y \end{array}\right\}$

$x = -27\frac{2}{3},\ y = 136\frac{5}{9}$

10. $\left\{\begin{array}{l} x + y = 18{\cdot}73 \\ 0{\cdot}56x + 13{\cdot}421y = 763{\cdot}4 \end{array}\right\}$

$x = -39{\cdot}8121\ldots,\ y = 58{\cdot}5421\ldots$

11. $\left\{\begin{array}{l} (x+5)(y+7) = (x+1)(y-9) + 112 \\ 2x + 10 = 3y + 1 \end{array}\right\}$

$x = 3,\ y = 5$

12. $\left\{\begin{array}{l} ax = by \\ x + y = c \end{array}\right\}$

$$x = \frac{bc}{a+b},\ y = \frac{ac}{a+b}$$

13. $\left\{\begin{array}{l} ax + by = c \\ fx + gy = h \end{array}\right\}$

$$x = \frac{cg - bh}{ag - bf},\ y = \frac{ah - cf}{ag - bf}$$

14. $\left\{\begin{array}{l} \dfrac{a}{b+y} = \dfrac{b}{3a+x} \\ ax + 2by = d \end{array}\right\}$

$$x = \frac{2b^2 - 6a^2 + d}{3a},\ y = \frac{3a^2 - b^2 + d}{3b}$$

15. $\left\{\begin{array}{l} bcx = cy - 2b \\ b^2y + \dfrac{a(c^3 - b^3)}{bc} = \dfrac{2b^3}{c} + c^3x \end{array}\right\}$

$$x = \frac{a}{bc},\ y = \frac{a+2b}{c}$$

16. $$\left\{\begin{array}{l} 3x + 5y = \dfrac{(8b - 2f)bf}{b^2 - f^2} \\ b^2x - \dfrac{bcf^2}{b+f} + (b+c+f)fy = f^2x + (b+2f)bf \end{array}\right\}$$

$$x = \frac{bf}{b-f},\quad y = \frac{bf}{b+f}$$

17. $$\left\{\begin{array}{l} x + y = 10 \\ x + z = 19 \\ y + z = 23 \end{array}\right\}$$

$$x = 3,\ y = 7,\ z = 16$$

18. $$\left\{\begin{array}{l} x + y + z = 29\frac{1}{4} \\ x + y - z = 18\frac{1}{4} \\ x - y + z = 13\frac{3}{4} \end{array}\right\}$$

$$x = 16,\ y = 7\tfrac{3}{4},\ z = 5\tfrac{1}{2}$$

19. $$\left\{\begin{array}{rl} x + y + z = & a \\ my \quad\ = & nx \\ pz \quad\ = & qx \end{array}\right\}$$

$$x = \frac{amp}{mp + np + mq},\ y = \frac{anp}{mp + np + mq},$$

$$z = \frac{amq}{mp + np + mq}$$

20. $$\left\{\begin{array}{l} 3x + 5y = 161 \\ 7x + 2z = 209 \\ 2y + z = 89 \end{array}\right\}$$

$$x = 17,\ y = 22,\ z = 45$$

21. $$\left\{\begin{array}{l} y + \frac{1}{2}x = 41 \\ x + \frac{1}{4}z = 20\frac{1}{2} \\ y + \frac{1}{5}z = 34 \end{array}\right\}$$

$$x = 18,\ y = 32,\ z = 10$$

22. $\left.\begin{matrix} ax + by = c \\ dx + ey = f \\ gy + hz = l \end{matrix}\right\}$

$$x = \frac{ce - bf}{ae - bd}, \; y = \frac{af - cd}{ae - bd}, \; z = \frac{a(el - fg) - d(bl - cg)}{h(ae - bd)}$$

23. $\left.\begin{matrix} 53 - \frac{1}{2}x - \frac{1}{2}z = y - 109 \\ \frac{1}{4}x + \frac{1}{6}y = 26 \\ 5y = 4z \end{matrix}\right\}$

$x = 64, \; y = 80, \; z = 100$

24. $\left.\begin{matrix} 2x - \frac{3}{4}y = 93 - \frac{1}{2}x - \frac{1}{4}y \\ 7x - 5z = y + x - 86 \\ \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 58 \end{matrix}\right\}$

$x = 48, \; y = 54, \; z = 64$

25. $\left.\begin{matrix} 3x - 100 = 5y + 360 \\ 2\frac{1}{3}x + 200 = 16\frac{1}{2}z - 610 \\ 2y + 3z = 548 \end{matrix}\right\}$

$x = 360, \; y = 124, \; z = 100$

26. $\left.\begin{matrix} 4x + 7y + 159 = 0 \\ 3\frac{1}{3}x = \frac{1}{6}z - 55 \\ 2x + y + 9z = 498 \end{matrix}\right\}$

$x = -13\frac{1}{2}, \; y = -15, \; z = 60$

27. $\left.\begin{matrix} 2x + 5y - 7z = -288 \\ 5x - y + 3z = 227 \\ 7x + 6y + z = 297 \end{matrix}\right\}$

$x = 13, \; y = 24, \; z = 62$

28. $\left.\begin{matrix} x + y + z = 30 \\ 8x + 4y + 2z = 50 \\ 27x + 9y + 3z = 64 \end{matrix}\right\}$

$x = \frac{2}{3}, \; y = -7, \; z = 36\frac{1}{3}$

29. $$\left.\begin{matrix} 18x - 7y - 5z = 11 \\ 4\frac{2}{3}y - \frac{2}{3}x + z = 108 \\ 3\frac{1}{2}z + 2y + \frac{3}{4}x = 80 \end{matrix}\right\}$$

$x = 12,\ y = 25,\ z = 6$

30. $$\left.\begin{matrix} ax + by + cz = h \\ a'x + b'y + c'z = h' \\ a''x + b''y + c''z = h'' \end{matrix}\right\}$$

$$x = \frac{hb'c'' - hb''c' + h'b''c - h'bc'' + h''bc' - h''b'c}{ab'c'' - ab''c' + a'b''c - a'bc'' + a''bc' - a''b'c}$$

$$y = \frac{ah'c'' - ah''c' + a'h''c - a'hc'' + a''hc' - a''h'c}{ab'c'' - ab''c' + a'b''c - a'bc'' + a''bc' - a''b'c}$$

$$z = \frac{ab'h'' - ab''h' + a'b''h - a'bh'' + a''bh' - a''b'h}{ab'c'' - ab''c' + a'b''c - a'bc'' + a''bc' - a''b'c}$$

31. $$\left.\begin{matrix} \frac{1}{x} + \frac{1}{y} = a \\ \frac{1}{x} + \frac{1}{z} = b \\ \frac{1}{y} + \frac{1}{z} = c \end{matrix}\right\}$$

$$x = \frac{2}{a+b-c},\ y = \frac{2}{a-b+c},\ z = \frac{2}{b+c-a}$$

32. $$\left.\begin{matrix} \frac{2}{x} - \frac{5}{3y} + \frac{1}{z} = 3\frac{4}{27} \\ \frac{1}{4x} + \frac{1}{y} + \frac{2}{z} = 6\frac{11}{72} \\ \frac{5}{6x} - \frac{1}{y} + \frac{4}{z} = 12\frac{1}{36} \end{matrix}\right\}$$

$x = 6, y = 9, z = \frac{1}{3}$

33. $$\left\{\begin{array}{l} \dfrac{2x+3y}{x+y} = 2\frac{1}{5} \\ \dfrac{x+z}{5(x-z)} = \frac{1}{3} \\ \dfrac{10x-3z}{4x-2z} = 2\frac{9}{14} \end{array}\right\}$$

$x = 16, y = 4, z = 4$

34. $$\left\{\begin{array}{l} ax+by = l \\ cx+du = m \\ ex+fz = n \\ gy+hz = p \end{array}\right\}$$

$$x = \frac{bhn+(gl-bp)f}{beh+afg}$$

$$y = \frac{afp+(el-an)h}{beh+afg}$$

$$z = \frac{bep+(an-el)g}{beh+afg}$$

$$u = \frac{bh(em-cn)+gf(am-cl)+bcfp}{d(beh+afg)}$$

35. $$\left\{\begin{array}{rl} x-9y+3z-10u = & 21 \\ 2x+7y-\ z-\ \ u = & 683 \\ 3x+\ y+5z+\ 2u = & 195 \\ 4x-6y-2z-\ 9u = & 516 \end{array}\right\}$$

$x = 100, y = 60, z = -13, u = -50$

36. $$\left\{\begin{array}{rl} x+\ \ y+\ \ z+\ u = & 1 \\ 16x+\ 8y+\ 4z+2u = & 9 \\ 81x+27y+\ 9z+3u = & 36 \\ 256x+64y+16z+4u = & 100 \end{array}\right\}$$

$x = \frac{1}{4}, y = \frac{1}{2}, z = \frac{1}{4}, u = 0$

37. $$\left\{\begin{aligned} \frac{x}{3}+\frac{y}{5}+\frac{2z}{7} &= 58 \\ \frac{5x}{4}+\frac{y}{6}+\frac{z}{3} &= 76 \\ \frac{x}{2}+\frac{3z}{8}+\frac{u}{5} &= 79 \\ y+z+u &= 248 \end{aligned}\right\}$$

$$x = 12,\ y = 30,\ z = 168,\ u = 50$$

38. $$\left\{\begin{aligned} \frac{xy}{ay+bx} &= l \\ \frac{yz}{cz+dy} &= m \\ \frac{xz}{ez+fx} &= n \end{aligned}\right\}$$

$$x = \frac{lmn(bde+acf)}{cfmn-bfln+bdlm}$$

$$y = \frac{lmn(bde+acf)}{afln+demn-adlm}$$

$$z = \frac{lmn(bde+acf)}{bcln-cemn+aclm}$$

39. $$\left\{\begin{aligned} x+y+z+t+u &= a \\ x+y+z+u+w &= b \\ x+y+z+t+w &= c \\ x+y+u+t+w &= d \\ x+z+u+t+w &= e \\ y+z+u+t+w &= f \end{aligned}\right\}$$

$$x = \frac{s}{5}-f,\ y = \frac{s}{5}-e,\ z = \frac{s}{5}-d,$$

$$u = \frac{s}{5}-c,\ t = \frac{s}{5}-b,\ w = \frac{s}{5}-a$$

$$(a+b+c+d+e+f = s)$$

(3) QUADRATIC EQUATIONS.

(a) With one Unknown Magnitude.

Formula.

$$x^2 + Px = Q$$

$$x = -\frac{P}{2} \pm \sqrt{\left(\frac{P^2}{4} + Q\right)}$$

Examples.

1. $ax^2 = b$

   $x = +\sqrt{\frac{b}{a}}, x = -\sqrt{\frac{b}{a}}$

2. $x^2 + 6x = 27$

   $x = 3, x = -9$

3. $x^2 - 7x + 3\frac{1}{4} = 0$

   $x = 6\frac{1}{2}, x = \frac{1}{2}$

4. $x^2 - 5\frac{3}{4}x = 18$

   $x = 8, x = -2\frac{1}{4}$

5. $3x^2 - 2x = 65$

   $x = 5, x = -4\frac{1}{3}$

6. $622x = 15x^2 + 6384$

   $x = 22\frac{4}{5}, x = 18\frac{2}{3}$

7. $20748 - 1616x + 21x^2 = 0$

   $x = 60\frac{2}{3}, x = 16\frac{2}{7}$

8. $9\frac{3}{5}x - 21\frac{15}{16} = x^2$

   $x = 5\frac{17}{20}, x = 3\frac{3}{4}$

9. $11\frac{3}{4}x - 3\frac{1}{2}x^2 = -41\frac{1}{4}$

$x = -2\frac{1}{7}$, $x = 5\frac{1}{2}$

10. $9\frac{1}{3}x^2 - 90\frac{1}{3}x + 195 = 0$

$x = 6\frac{5}{7}$, $x = 3\frac{1}{4}$

11. $\frac{18x^2}{5} + \frac{18078x}{65} + 4728 = 0$

$x = -25\frac{10}{39}$, $x = -52$

12. $x^2 - 8x = 14$

$x = 4 + \sqrt{30}$, $x = 4 - \sqrt{30}$

Or $x = 9{,}4772\ldots$, $x = -1{,}4772\ldots$

13. $3x^2 + x = 7$

$$x = \frac{-1+\sqrt{85}}{6}, \quad x = \frac{-1-\sqrt{85}}{6}$$

Or $x = 1{\cdot}3699\ldots$, $x = -1{\cdot}7032$

14. $118x - 2\frac{1}{2}x^2 = 20$

$$x = \frac{118+\sqrt{13724}}{5}, \quad x = \frac{118-\sqrt{13724}}{5}$$

Or $x = 47{\cdot}0298\ldots$, $x = 0{\cdot}1701\ldots$

15. $6x - 30 = 3x^2$

$x = 1 + \sqrt{-9}$, $x = 1 - \sqrt{-9}$

16. $8x^2 - 7x + 34 = 0$

$$x = \frac{7+\sqrt{-1039}}{16}, \quad x = \frac{7-\sqrt{-1039}}{16}$$

17. $4x^2 - 9x = 5x^2 - 255\frac{3}{4} - 8x$

$x = 15\frac{1}{2}$, $x = -16\frac{1}{2}$

18. $80x + \frac{3x^2}{4} + \frac{21x-27782}{12} = 1859\frac{1}{3} - 3x^2$

$x = -46$, $x = 24\frac{1}{5}$

19. $\dfrac{x}{x+60} = \dfrac{7}{3x-5}$

$x = 14,\ x = -10$

20. $\dfrac{40}{x-5} + \dfrac{27}{x} = 13$

$x = 9,\ x = 1\frac{7}{13}$

21. $\dfrac{8x}{x+2} - 6 = \dfrac{20}{3x}$

$x = 10,\ x = -\frac{2}{3}$

22. $\dfrac{48}{x+3} = \dfrac{165}{x+10} - 5$

$x = 5\frac{2}{3},\ x = 5$

23. $\dfrac{31}{6x} = \dfrac{16}{117-2x} + 1$

$x = 67\frac{1}{6},\ x = 4\frac{1}{2}$

24. $\dfrac{2x+3}{10-x} = \dfrac{2x}{25-3x} - 6\frac{1}{2}$

$x = 13\frac{22}{31},\ x = 8$

25. $\dfrac{25x+180}{10x-81} = \dfrac{40x}{5x-8} - \frac{5}{5}$

$x = 14\frac{5}{5},\ x = \frac{72}{249}$

26. $\dfrac{18+x}{6(3-x)} = \dfrac{20x+9}{19-7x} - \dfrac{65}{4(3-x)}$

$x = 7\frac{22}{113},\ x = 2\frac{1}{2}$

27. $adx - acx^2 = bcx - bd$

$x = \dfrac{d}{c},\ x = -\dfrac{b}{a}$ correct

28. $\dfrac{a^2x^2}{f^2} - \dfrac{2ax}{g} + \dfrac{f^2}{g^2} = 0$

$$x = \frac{f^2}{ag},\ x = \frac{f^2}{ag}$$

29. $abx^2 + \dfrac{3a^2x}{c} = \dfrac{6a^2 + ab - 2b^2}{c^2} - \dfrac{b^2x}{c}$

$$x = \frac{2a-b}{ac},\ x = -\frac{3a+2b}{bc}$$

30. $\dfrac{2c^2}{d^2} + \dfrac{ac}{d} - (a - b)(2c + ad)\dfrac{x}{d} = (a + b)\dfrac{cx}{d} - (a^2 - b^2)x^2$

$$x = \frac{2c + ad}{d(a+b)},\ x = \frac{c}{d(a-b)}$$

31. $32a^{2m}c^{n-1} + 4a^{m+3}c^{n-1}(ac^3 - 2)x = a^7c^{n+2}x^2$

$$x = 4a^{m-3},\ x = -\frac{8a^{m-4}}{c^3}$$

32. $cx + \dfrac{ac}{a+b} = (a + b)x^2$

$$x = \frac{c + \sqrt{(c^2+4ac)}}{2(a+b)},\ x = \frac{c - \sqrt{(c^2+4ac)}}{2(a+b)}$$

33. $9a^4b^4x^2 - 6a^3b^2x - b^2 = 0$

$$x = \frac{a + \sqrt{(a^2+b^2)}}{3a^2b^2},\ x = \frac{a - \sqrt{(a^2+b^2)}}{3a^2b^2}$$

34. $abx^2 - 2x(a + b)\sqrt{ab} = (a - b)^2$

$$x = \frac{a + b \pm \sqrt{(2a^2 + 2b^2)}}{\sqrt{ab}}$$

35. $ax^2 + b^2 + c^2 = a^2 + 2bc + 2(b - c)x\sqrt{a}$

$$x = \frac{b - c + a}{\sqrt{a}},\ x = \frac{b - c - a}{\sqrt{a}}$$

36. $cx^2 - 2cx\sqrt{d} = dx^2 - cd$

$$x = \frac{\sqrt{cd}}{\sqrt{c}+\sqrt{d}}, \quad x = \frac{\sqrt{cd}}{\sqrt{c}-\sqrt{d}}$$

✓ 37. $(4a^2 - 9cd^2)x^2 + (4a^2c^2 + 4abd^2)x + (ac^2 + bd^2)^2 = 0$

$$x = -\frac{ac^2+bd^2}{2a+3d\sqrt{c}}, \quad x = -\frac{ac^2+bd^2}{2a-3d\sqrt{c}}$$

38. $ab^3x^2 + (1 + c)bd\sqrt{c} + cb^2x^2 =$
$[b^3d\sqrt{c} + (ab + c)(1 + c)]x$

$$x = \frac{bd\sqrt{c}}{ab+c}, \quad x = \frac{1+c}{b^2}$$

39. $\frac{5a + 10ab^2}{9b^2 - 3a^2b^2}x^2 - \left(\frac{5\sqrt{(a+b)}}{3b^3} + \frac{(1+2b^2)cd\sqrt{c}}{3-a^2}\right)x$
$+ \frac{cd}{ab}\sqrt{(a+b)c} = 0$

$$x = \frac{(3-a^2)\sqrt{(a+b)}}{ab(1+2b^2)}, \quad x = \frac{3b^2cd\sqrt{c}}{5a}$$

40. $ax = b + \sqrt{cx}$

$$x = \frac{2ab + c + \sqrt{(4abc + c^2)}}{2a^2} *$$

41. $3\sqrt{(112 - 8x)} = 19 + \sqrt{(3x + 7)}$

$$x = 6$$

---

* Only this one value can be here used; the other obtains in $ax = b - \sqrt{cx}$. Both of them produce the Equation $a^2x^2 - (2ab+c)\times x + b^2 = 0$. These observations apply also to Equations 41, 42, 43, 44.

42. $\sqrt{(2x+7)} + \sqrt{(3x-18)} = \sqrt{(7x+1)}$

$x = 9$

43. $5\sqrt{(62+3x)} - \frac{1}{2}\sqrt{(95\frac{2}{3} - 5x)} = 41$

$x = 6\frac{1}{3}$

44. $7\sqrt{(\frac{3}{2}x-5)} - \sqrt{\left(\frac{x}{5} + 45\right)} - \frac{7}{4}\sqrt{(10x+56)} = 0$

$x = 20$

45. $ax^{2n} + bx^n = c$

$$x = \sqrt[n]{\frac{-b + \sqrt{(b^2+4ac)}}{2a}},$$

$$x = \sqrt[n]{\frac{-b - \sqrt{(b^2+4ac)}}{2a}}$$

46. $x^4 - 74x^2 = -1225$

$x = \pm 5,\ x = \pm 7$

47. $3x^6 + 42x^3 = 3321$

$x = 3,\ x = -\sqrt[3]{41}$

(b) With more Unknown Magnitudes.

1. $\left\{\begin{matrix} x^2 + y^2 = a \\ x^2 - y^2 = b \end{matrix}\right\}$

$$x = \pm\sqrt{\frac{a+b}{2}},\ y = \pm\sqrt{\frac{a-b}{2}}$$

2. $\left\{\begin{matrix} x + y = a \\ xy = b \end{matrix}\right\}$

$$x = \frac{a + \sqrt{(a^2-4b)}}{2},\ y = \frac{a - \sqrt{(a^2-4b)}}{2}$$

$$\text{Or } x = \frac{a - \sqrt{(a^2-4b)}}{2},\ y = \frac{a + \sqrt{(a^2-4b)}}{2}$$

3. $\left.\begin{matrix} x+y=a \\ x^2+y^2=b \end{matrix}\right\}$

$$x=\frac{a+\sqrt{(2b-a^2)}}{2}, y=\frac{a-\sqrt{(2b-a^2)}}{2}$$

Or $$x=\frac{a-\sqrt{(2b-a^2)}}{2}, y=\frac{a+\sqrt{(2b-a^2)}}{2}$$

4. $\left.\begin{matrix} xy=a \\ x^2+y^2=b \end{matrix}\right\}$

$$x=\pm\sqrt{\frac{b+\sqrt{(b^2-4a^2)}}{2}}$$

$$y=\pm\sqrt{\frac{b-\sqrt{(b^2-4a^2)}}{2}}$$

Or $$x=\pm\sqrt{\frac{b-\sqrt{(b^2-4a^2)}}{2}}$$

$$y=\pm\sqrt{\frac{b+\sqrt{(b^2-4a^2)}}{2}}$$

5. $\left.\begin{matrix} x+y=a \\ x^3+y^3=b \end{matrix}\right\}$

$$x=\frac{a}{2}+\sqrt{\frac{4b-a^3}{12a}}, y=\frac{a}{2}-\sqrt{\frac{4b-a^3}{12a}}$$

Or $$x=\frac{a}{2}-\sqrt{\frac{4b-a^3}{12a}}, y=\frac{a}{2}+\sqrt{\frac{4b-a^3}{12a}}$$

6. $\left.\begin{matrix} 2x+3y=118 \\ 5x^2-7y^2=4333 \end{matrix}\right\}$

$$x=35,\ y=16$$

Or $$x=-229\tfrac{6}{17},\ y=192\tfrac{4}{17}$$

7. $\left.\begin{matrix} ax+by=h \\ cx^2+dy^2=k \end{matrix}\right\}$

$$x=\frac{adh\pm b\sqrt{(a^2dk-cdh^2+b^2ck)}}{a^2d+b^2c}$$

$$y=\frac{bch\pm a\sqrt{(a^2dk-cdh^2+b^2ck)}}{a^2d+b^2c}$$

8. $\left\{\begin{array}{l} x^2-y^2=h \\ (x+y+a)^2+(x-y+a)^2=k \end{array}\right\}$

$$x=\frac{-a\pm\sqrt{(2h+k-a^2)}}{2}$$

$$y=\pm\sqrt{\frac{\frac{1}{2}k-h\mp a\sqrt{(2h+k-a^2}}{2}}$$

9. $\left\{\begin{array}{l} \dfrac{18x}{y}=\dfrac{8y}{x} \\ 3xy+2x+y=485 \end{array}\right\}$

$x=10,\ y=15$

Or $x=-10\frac{7}{9},\ y=-16\frac{1}{6}$*

10. $\left\{\begin{array}{l} \dfrac{ax}{y}=\dfrac{by}{x} \\ cxy+dx+ey=h \end{array}\right\}$

† $\left\{\begin{array}{l} x=\dfrac{-(e\sqrt{a}+d\sqrt{b})\pm\sqrt{[(e\sqrt{a}+d\sqrt{b})^2+4ch\sqrt{ab}]}}{2c\sqrt{a}} \\ y=\dfrac{-(e\sqrt{a}+d\sqrt{b})\pm\sqrt{[(e\sqrt{a}+d\sqrt{b})^2+4ch\sqrt{ab}]}}{2c\sqrt{b}} \end{array}\right\}$

Or $\left\{\begin{array}{l} x=\dfrac{-(e\sqrt{a}-d\sqrt{b})\pm\sqrt{[(e\sqrt{a}-d\sqrt{b})^2-4ch\sqrt{ab}]}}{2c\sqrt{a}} \\ y=\dfrac{+(e\sqrt{a}-d\sqrt{b})\mp\sqrt{[(e\sqrt{a}-d\sqrt{b})^2-4ch\sqrt{ab}]}}{2c\sqrt{b}} \end{array}\right\}$

---

* These equations admit of two more solutions, viz.

$$x=\frac{1+\sqrt{-34919}}{18},\ y=\frac{-1-\sqrt{-34919}}{12},$$

$$x=\frac{1-\sqrt{-34919}}{18},\ y=\frac{-1+\sqrt{-34919}}{12},$$

which, however, are imaginary.

† These equations give 4 values each to x and y, corresponding in pairs.

11. $\left\{\begin{matrix} x+y+x^2+y^2=a \\ x-y+x^2-y^2=b \end{matrix}\right\}$

$$x=\frac{-1\pm\sqrt{(2a+2b+1)}}{2}$$

$$y=\frac{-1\pm\sqrt{(2a-2b+1)}}{2}$$

Or $$x=\frac{-1\pm\sqrt{(2a+2b+1)}}{2}$$

$$y=\frac{-1\pm\sqrt{(2a-2b+1)}}{2}$$ *

12. $\left\{\begin{matrix} x+y=xy \\ x+y+x^2+y^2=a \end{matrix}\right\}$

$$x=\frac{1\pm\sqrt{(4a+1)}+\sqrt{[4a-6\mp6\sqrt{(4a+1)}]}}{4}$$

$$y=\frac{1\pm\sqrt{(4a+1)}-\sqrt{[4a-6\mp6\sqrt{(4a+1)}]}}{4}$$ †

13. $\left\{\begin{matrix} ax-by=g \\ a^3x^3-b^3y^3=hxy \end{matrix}\right\}$

$$x=\frac{g}{2a}\left(1\pm\sqrt{\frac{h+abg}{h-3abg}}\right), y=\frac{g}{2b}\left(-1\pm\sqrt{\frac{h+abg}{h-3abg}}\right)$$

14. $\left\{\begin{matrix} (x-y)(x^2-y^2)=a \\ (x+y)(x^2+y^2)=b \end{matrix}\right\}$

$$x=\frac{\sqrt{(2b-a)}\pm\sqrt{a}}{2\sqrt[6]{(2b-a)}}, y=\frac{\sqrt{(2b-a)}\mp\sqrt{a}}{2\sqrt[6]{(2b-a)}}$$

---

* These Equations give 4 pairs of values to x and y.

† These values of x and y may also be interchanged.

15. $\left.\begin{cases} x^2+y^2+z^2=a \\ y^2=2xz+b \\ cx=dz \end{cases}\right\}$

$$x=\frac{d\sqrt{(a-b)}}{c+d}, y=\frac{\sqrt{[2acd+b(c^2+d^2)]}}{c+d},$$

$$z=\frac{c\sqrt{(a-b)}}{c+d}$$

16. $\left.\begin{cases} x(y+z)=a \\ y(x+z)=b \\ z(x+y)=c \end{cases}\right\}$

$$x=\pm\sqrt{\frac{(a-c+b)(a-b+c)}{2(c-a+b)}}$$

$$y=\pm\sqrt{\frac{(a-c+b)(c-a+b)}{2(a-b+c)}}$$

$$x=\pm\sqrt{\frac{(c-a+b)(a-b+c)}{2(a-c+b)}}$$

17. $\left\{\begin{array}{l} \dfrac{xyz}{x+y}=a \\ \dfrac{xyz}{y+z}=b \\ \dfrac{xyz}{x+z}=c \end{array}\right\}$

$$x=\pm\sqrt{\frac{2abc(ab-ac+bc)}{(ab+ac-bc)\ (bc+ac-ab)}}$$

$$y=\pm\sqrt{\frac{2abc(bc+ac-ab)}{(ab+ac-bc)\ (ab-ac+bc)}}$$

$$z=\pm\sqrt{\frac{2abc(ab+ac-bc)}{(ab-ac+bc)\ (bc+ac-ab)}}$$

18. $\left\{\begin{array}{l} xy = p \\ (b-y)\,z = p' \\ (a-x)\,(c-z) = p'' \end{array}\right\}$

$$x = \frac{-A \pm \sqrt{[A^2 - 4p\,(p'-bc)\,(p''-ac)]}}{2\,(p' - bc)}$$

$$y = \frac{-A \mp \sqrt{[A^2 - 4p\,(p'-bc)\,(p''-ac)]}}{2\,(p'' - ac)}$$

$$z = \frac{-B \mp \sqrt{[B^2 - 4p'\,(p-ab)\,(p''-ac)]}}{2\,(p - ab)}$$

$(cp - ap' - bp'' + abc = A,\ cp - ap' + bp'' - abc = B)$

19. $\left\{\begin{array}{l} x^2 + y^2 + z^2 = k \\ ax + a'y + a''z = 0 \\ bx + b'y + b''z = 0 \end{array}\right\}$

$$x = (a'b'' - a''b')\,A,\ y = (a''b - ab'')\,A,$$
$$z = (ab' - a'b)\,A$$

$$\pm \sqrt{\frac{k}{(ab'-a'b)^2 + (a'b''-a''b')^2 + (a''b-ab'')^2}} = A.$$

20. $\left\{\begin{array}{l} axy + bx + cy + d = 0 \\ a'yz + b'y + c'z + d' = 0 \\ a''zx + b''z + c'x + d = 0 \end{array}\right\}$

The elimination of y, z, gives a Quadratic Equation,

$$(a''x + b'')\,[(bb' - ad')\,x + b'd - cd'] +$$
$$(c''x + d'')\,[(ac' - a'b)\,x + cc' - a'd] = 0$$

If x be hence found, we easily get y and z.

(The two last kinds of Equations will be found generalized in p. 156.)

(4) SOLUTION OF EQUATIONS OF THE HIGHER DEGREES.

(a) Cardan's Formula.

$$x^3 = Px + Q$$

$$x = \sqrt[3]{\frac{Q + \sqrt{\left(Q^2 - \frac{4P^3}{27}\right)}}{2}} + \sqrt[3]{\frac{Q - \sqrt{\left(Q^2 - \frac{4P^3}{27}\right)}}{2}}$$

Examples.

1. $x^3 - 3x - 2 = 0$

   $x = 2$

2. $x^3 + 12x + 63 = 0$

   $x = -3$

3. $x^3 - 21x + 344 = 0$

   $x = -8$

4. $x^3 - 6x - 40 = 0$

   $x = \sqrt[3]{(20 + \sqrt{392})} + \sqrt[3]{(20 - \sqrt{392})} =$

   $\sqrt[3]{(2 + \sqrt{2})^3} + \sqrt[3]{(2 - \sqrt{2})^3} = 4$

5. $x^3 + 3x - 14 = 0$

   $x = \sqrt[3]{(7 + \sqrt{50})} + \sqrt[3]{(7 - \sqrt{50})} =$

   $\sqrt[3]{(1 + \sqrt{2})^3} + \sqrt[3]{(1 - \sqrt{2})^3} = 2$

6. $x^3 - \frac{15}{2}x + 290\frac{1}{2} = 0$

   $$x = \sqrt[3]{\frac{-581 + \sqrt{337311}}{4}} + \sqrt[3]{\frac{-581 - \sqrt{337311}}{4}}$$

   $$= \sqrt[3]{\left(\frac{-7 + \sqrt{39}}{2}\right)^3} + \sqrt[3]{\left(\frac{-7 - \sqrt{39}}{2}\right)^3} = -7$$

7. $x^3 - 12x^2 + 57x - 94 = 0$

$$x = 4 + \sqrt[3]{3} - \sqrt[3]{9} = 3{\cdot}36216\ldots$$

8. $x^3 - 12x - 28 = 0$

$$x = \sqrt[3]{(14 + \sqrt{132})} + \sqrt[3]{(14 - \sqrt{132})} = 4{\cdot}30213\ldots$$

9. $x^3 + 6x^2 + 20x + 15 = 0$

$$x = -2 + \sqrt[3]{\frac{81 + \sqrt{12705}}{18}} + \sqrt[3]{\frac{81 - \sqrt{12705}}{18}}$$

$$= -2 + \sqrt[3]{\left(\frac{3 + \sqrt{105}}{6}\right)^3} + \sqrt[3]{\left(\frac{3 - \sqrt{105}}{6}\right)^3} = -1$$

10. $x^3 - 15x^2 + 71x - 297 = 0$

$$x = 5 + \sqrt[3]{\frac{864 + \sqrt{746304}}{3}} + \sqrt[3]{\frac{864 - \sqrt{746304}}{3}}$$

$$= 5 + \sqrt[3]{\left(\frac{9 + \sqrt{69}}{3}\right)^3} + \sqrt[3]{\left(\frac{9 - \sqrt{69}}{3}\right)^3} = 11$$

11. $x^3 - 12x^2 + 36x - 7 = 0$

$$x = 4 + \sqrt[3]{\frac{-9 + \sqrt{-175}}{2}} + \sqrt[3]{\frac{-9 - \sqrt{-175}}{2}}$$

$$= 4 + \sqrt[3]{\left(\frac{3 + \sqrt{-7}}{2}\right)^3} + \sqrt[3]{\left(\frac{3 - \sqrt{-7}}{2}\right)^3} = 7$$

(b) By finding their Rational Roots.

1. $x^3 - 9x^2 + 26x - 24 = 0$

   2, 3, 4

2. $x^3 - 8x^2 + 5x + 14 = 0$

   -1, 2, 7

3. $x^3 - 49x - 120 = 0$

   -3, -5, 8

4. $x^3 - 18x^2 + 87x - 110 = 0$

   2, 5, 11

5. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$

$1, 2, 3, 4$

6. $x^4 - 45x^2 - 40x + 84 = 0$

$1, -2, -6, 7$

7. $x^4 + 29x^3 + 287x^2 + 1147x + 1560 = 0$

$-3, -5, -8, -13$

8. $x^3 + \frac{17}{4}x^2 - \frac{79}{8}x + \frac{15}{4} = 0$

$\frac{1}{2}, \frac{5}{4}, -6$

9. $x^3 - \frac{13}{12}x^2 + \frac{5}{8}x - \frac{1}{24} = 0$

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

10. $x^3 - \frac{14}{3}x^2 + 7x - \frac{10}{3} = 0$

$1, \frac{5}{3}, 2$

11. $x^3 + \frac{3}{5}x^2 + \frac{9}{100}x - \frac{9}{100} = 0$

$\frac{1}{5}, -\frac{3}{10}, -\frac{3}{2}$

12. $x^3 - \frac{39}{28}x^2 + \frac{31}{56}x - \frac{3}{56} = 0$

$\frac{1}{2}, \frac{1}{7}, \frac{3}{4}$

13. $x^3 - \frac{139}{24}x^2 + \frac{529}{48}x + \frac{55}{16} = 0$

$-\frac{5}{8}, \frac{5}{2}, \frac{11}{3}$

14. $x^3 + \frac{82}{15}x^2 - \frac{173}{5}x - \frac{126}{5} = 0$

$-\frac{2}{3}, \frac{21}{5}, -9$

15. $x^4 - \frac{19}{4}x^3 + \frac{49}{8}x^2 - \frac{11}{4}x + \frac{3}{8} = 0$

$\frac{1}{2}, \frac{1}{4}, 1, 3$

16. $x^4 - \frac{41}{8}x^3 + \frac{287}{32}x^2 - \frac{393}{64}x + \frac{45}{32} = 0$

$\frac{1}{2}, \frac{3}{4}, \frac{15}{8}, 2$

17. $x^3 - 14x^2 - 5x + 70 = 0$

$14, +\sqrt{5}, -\sqrt{5}$

18. $x^3 - 13x^2 + 49x - 45 = 0$

$5, 4 + \sqrt{7}, 4 - \sqrt{7}$

19. $x^3 - 13x^2 + 38x + 16 = 0$

$8, \frac{5}{2} + \frac{1}{2}\sqrt{33}, \frac{5}{2} - \frac{1}{2}\sqrt{33}$

20. $x^3 - 6x^2 + 19x - 44 = 0$

$4, 1 + \sqrt{-10}, 1 - \sqrt{-10}$

21. $x^3 + \frac{7}{6}x^2 + \frac{13}{2}x + \frac{21}{2} = 0$

$-\frac{3}{2}, \frac{1}{6} + \frac{1}{6}\sqrt{-251}, \frac{1}{6} - \frac{1}{6}\sqrt{-251}$

22. $x^4 + x^3 - 24x^2 + 43x - 21 = 0$

$1, 3, -\frac{5}{2} + \frac{1}{2}\sqrt{53}, -\frac{5}{2} - \frac{1}{2}\sqrt{53}$

23. $x^5 - 3x^4 - 8x^3 + 24x^2 - 9x + 27 = 0$

$3, 3, -3, +\sqrt{-1}, -\sqrt{-1}$

24. $x^5 - \frac{3}{2}x^4 - 6x^3 + 9x^2 - 13x + 19\frac{1}{2} = 0$

$\frac{3}{2}, +\sqrt{(3 + \sqrt{22})}, -\sqrt{(3 + \sqrt{22})},$
$+\sqrt{(3 - \sqrt{22})}, -\sqrt{(3 - \sqrt{22})}$

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(5) TWO GENERAL CASES OF EQUATIONS WITH MORE THAN TWO UNKNOWN MAGNITUDES WHICH ADMIT OF SOLUTION.

1. Let x', x'', $\ldots\ldots x^{n'}$ denote n unknown quantities. Suppose we have an equation of the form

(a) $a'x'^m + a''x''^m + a'''x'''^m + a''''x''''^m + \ldots\ldots$
$+ a^{n'}(x^{n'})^m = K$

and also $n - 1$ equations of the following form, in which the unknown quantities do not exceed the first degree:

$$b'x' + b''x'' + b'''x''' + b''''x'''' + \ldots\ldots + b^{n'}x^{n'} = 0$$
$$c'x' + c''x'' + c'''x''' + c''''x'''' + \ldots\ldots + c^{n'}x^{n'} = 0$$
$$d'x' + d''x'' + d'''x''' + d''''x'''' + \ldots\ldots + d^{n'}x^{n'} = 0$$
&c.

then we can express the quantities sought, by the solution of these last equations, in terms of one of them thus: make $x'' = A''x'$, $x''' = A'''x'$, &c.... $x^{n'} = A^{n'}x'$ where A'', A''', &c. are known quantities; which being substituted in the first equation give

$$x' = \sqrt[m]{\frac{K}{a' + a''A''^m + a'''A''''^m + a''''A''''^m \ldots\ldots + a^{n'}(A^{n'})^m}}$$

whence are obtained the values of x'', x''' &c. This admits of being still further generalized.

II. Let there be the following n recurring Equations between x', x'' &c.;

$$a'x'x'' + b'x' + c'x'' + d' = 0$$
$$a''x''x''' + b''x'' + c''x''' + d'' = 0$$
$$a'''x'''x'''' + b'''x''' + c'''x'''' + d''' = 0$$
.....
$$a^{(n-1)'}x^{(n-1)'}x^{n'} + b^{(n-1)'}x^{(n-1)'} + c^{(n-1)'}x^{n'} + d^{(n-1)'} = 0$$
$$a^{n'}x^{n'}x' + b^{n'}x^{n'} + c^{n'}x' + d^{n'} = 0$$

From the first equation find x'' in terms of x', which substitute in the second equation. Find also x''' from the second in terms of x', and so on throughout, until at length is obtained an equation of this form

$$x^{n'} = \frac{Ax' + B}{Cx' + D}$$

If this value be substituted in the last equation there results a quadratic, whereby x' may be found. Hence may be found the other unknown quantities.

If the roots of an equation be given, the equation itself may be found. How? What is the equation, for example, whose roots are 1, 3, -1, -4? What is that whose roots are 6, $2+3\sqrt{-1}$, $2-3\sqrt{-1}$? The coefficients, it appears, bear a certain relation to the roots. What is this affinity? When the equations

$$\left.\begin{aligned} x+y+z&=a\\ xy+xz+yz&=b\\ xyz&=c \end{aligned}\right\}$$

Or when

$$\left.\begin{aligned} x+y+z+w&=a\\ xy+xz+xw+yz+yw+zw&=b\\ xyz+xyw+xzw+yzw&=c\\ xyzw&=d \end{aligned}\right\}$$

are given; then by elimination we obtain one of the unknown quantities in the form of an equation of the 3rd or 4th degree respectively. What will be the forms of these equations? When the equation is of an odd number of dimensions it has at least one real root; why? When $h+k\sqrt{-1}$ is any root of an equation, $h-k\sqrt{-1}$ is also a root. How is this proved? Let it be granted (it is easily demonstrated) that all imaginary expressions are reducible to the form $h+k\sqrt{-1}$; how many imaginary and real roots can an equation of the nth degree have, when n is even or odd? When *Cardan's Rule* gives an imaginary result, has the equation no real root?

XIV. SOLUTION OF EQUATIONS BY APPROXIMATION.

(1) EQUATIONS WITH ONE UNKNOWN MAGNITUDE.

First method.

LET $X = 0$ be an equation involving x. Also let w be a value of x found by trial, which differs from the real value by something less than unity. Then for x put $w+h$, and retain only those terms in the result which have the first power of h, and we have an equation of the form $A+Bh=0$, A and B being known quantities. Hence obtaining h, and therefore $x=w+h$, we have a second approximation of x. Substitute this value increased by h for x in the given equation, and proceed to find h as before, and so on continually, to any required degree of approximation.

By the application of this principle to the general equation

$$x^m + ax^{m-1} + bx^{m-2} + cx^{m-3} + \ldots\ldots + hx^2 + kx + l = 0$$

we obtain the following expressions for the approximate value:

$$\frac{(m-1)w^m + (m-2)aw^{m-1} + (m-3)bw^{m-2} + \ldots + 1hw^2 - l}{mw^{m-1} + (m-1)aw^{m-2} + (m-2)bw^{m-3} + \ldots + 2hw + k}$$

For the Cubic Equation $x^3 + ax^2 + bx + c = 0$, we have

$$x = \frac{2w^3 + aw^2 - c}{3w^2 + 2aw + b}$$

For the ~~Quadratic~~ Biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$, we get

$$x = \frac{3w^4 + 2aw^3 + bw^2 - d}{4w^3 + 3aw^2 + 2bw + c}$$

For $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$, we find

$$x = \frac{4w^5 + 3aw^4 + 2bw^3 + cw^2 - e}{5w^4 + 4aw^3 + 3bw^2 + 2cw + d}$$

and so on.

Examples.

1. $x^3 = 2$

   $w = 1$, $w' = \frac{4}{3}$, $w'' = \frac{91}{72}$, $w''' = \frac{2255638}{1788096}$, &c.*

2. $x^3 = 30$

   $w = 3$, $w' = \frac{28}{9}$, $w'' = \frac{65774}{21168}$, &c.

3. $x^3 - 12x^2 + 57x - 94 = 0$

   $w = 3$, $w' = \frac{10}{3}$, $w'' = \frac{938}{279}$, &c.

4. $x^3 - 15x^2 + 72x - 109 = 0$

   $w = 5$, $w' = \frac{16}{3}$, $w'' = \frac{385}{72}$, &c.

5. $x^3 - 13x^2 + 38x + 17 = 0$

   $w = 8$, $w' = \frac{175}{22}$, $w'' = \frac{1778984}{225674}$, &c.

---

* w', w'' &c. denote the successive approximating values of x. There is no need to calculate more than one root.

6. $x^3 + 2x^2 + 3x - 52 = 0$

$w = 3,\ w' = \frac{62}{21},\ w'' = \frac{1119676}{379323}$, &c.

7. $x^3 - 12x - 132 = 0$

$w = 6,\ w' = \frac{47}{8},\ w'' = \frac{137615}{23436}$, &c.

8. $x^4 - 4x^3 + 18 = 0$

$w = 2,\ w' = \frac{17}{8},\ w'' = \frac{137597}{64736}$, &c.

9. $x^4 + 8x^2 + 16x - 440 = 0$

$w = 4,\ w' = \frac{167}{42},\ w'' = \frac{40961047771}{10502040036}$, &c.

Since the third approximation does not probably give the root true to more than three decimal places, it will be useless to develope the fractions beyond that term.

Second Method.

Let $X = 0$ be an equation in x. Its roots may be found by Continual Fractions. According to the general principle laid down in VIII, we proceed as follows:

In $X = 0$, for x, put $a + \frac{1}{x^I}$, and the result is of the form $X^I = 0$, involving x^I. Then let a^I, $a^I + 1$ be integer limits of the root of this equation, and for x^I put $a^I + \frac{1}{x^{II}}$ in $X^I = 0$, and the result will be of the form $X^{II} = 0$ a function of x^{II}. Find a^{II}, $a^{II} + 1$, limits to x^{II} as before, and so on repeating the operation, until there results

$$x = a + \frac{1}{x^I} = a + \frac{1}{a^I + \frac{1}{x^{II}}} = a + \frac{1}{a^I + \frac{1}{a^{II} + \text{&c.}}}$$

which being reduced will give the several approximating values of x.

EXAMPLES.

$$X = x^3 - 2 = 0 \qquad x = 1 + \frac{1}{x^I}$$

$$X^I = x^{I3} - 3x^{I2} - 3x^I - 1 = 0 \qquad x^I = 3 + \frac{1}{x^{II}}$$

$$X^{II} = 10x^{II3} - 6x^{II2} - x^{II} - 1 = 0 \qquad x^{II} = 3 + \frac{1}{x^{III}}$$

$$X^{III} = 3x^{III3} - 12x^{III2} - 24x^{III} - 10 = 0 \qquad x^{III} = 5 + \frac{1}{x^{IV}}$$

$$X^{IV} = 55x^{IV3} - 81x^{IV2} - 33x^{IV} - 3 = 0 \qquad x^{IV} = 1 + \frac{1}{x^V}$$

$$X^V = 62x^{V3} + 30x^{V2} - 84x^V - 55 = 0 \qquad x^V = 1 + \frac{1}{x^{VI}}$$

&c.

Approximating value : 1, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{29}{23}$, $\frac{34}{27}$, $\frac{13}{30}$, &c.

The root : 1·25992……

2. $x^3 - 15x^2 + 63x - 50 = 0$

……

$$X = x^3 - 15x^2 + 63x - 50 = 0 \qquad x = 1 + \frac{1}{x^I}$$

$$X^I = x^{I3} - 36x^{I2} + 12x^I - 1 = 0 \qquad x^I = 35 + \frac{1}{x^{II}}$$

$$X^{II} = 806x^{II3} - 1167x^{II2} - 69x^{II} - 1 = 0 \qquad x^{II} = 1 + \frac{1}{x^{III}}$$

$$X^{III} = 431x^{III3} - 15x^{III2} - 1251x^{III} - 806 = 0 \qquad x^{III} = 1 + \frac{1}{x^{IV}}$$

&c.

Approximating values : 1, $\frac{36}{35}$, $\frac{37}{36}$, $\frac{73}{71}$, &c.

The root : 1·02803……

3. $x^3 - 12x^2 + 45x - 53 = 0$

.....

$X = x^3 - 12x^2 + 45x - 53 = 0$	$x = 5 + \frac{1}{x^I}$
$X^I = 3x^{I3} - 3x^I - 1 = 0$	$x^I = 1 + \frac{1}{x^{II}}$
$X^{II} = x^{II3} - 6x^{II2} - 9x^{II} - 3 = 0$	$x^{II} = 7 + \frac{1}{x^{III}}$
$X^{III} = 17x^{III3} - 54x^{III2} - 15x^{III} - 1 = 0$	$x^{III} = 3 + \frac{1}{x^{IV}}$
$X^{IV} = 73x^{IV3} - 120x^{IV2} - 99x^{IV} - 17 = 0$	$x^{IV} = 3 + \frac{1}{x^V}$
$X^V = 111x^{V3} - 297x^{V2} - 318x^V - 73 = 0$	$x^V = 3 + \frac{1}{x^{VI}}$
$X^{VI} = 703x^{VI3} - 579x^{VI2} - 702x^{VI} - 111 = 0$	$x^{VI} = 1 + \frac{1}{x^{VII}}$
&c.	

Approximating values : 5, 6, $\frac{47}{8}$, $\frac{147}{25}$, $\frac{541}{38}$, $\frac{1170}{199}$, $\frac{1511}{257}$, &c.
The root : 5·879385.....

4. $x^3 - 12x^2 + 57x - 94 = 0$

.....

$X = x^3 - 12x^2 + 57x - 94 = 0$	$x = 3 + \frac{1}{x^I}$
$X^I = 4x^{I3} - 12x^{I2} + 3x^I - 1 = 0$	$x^I = 3 - \frac{1}{x^{II}}$
$X^{II} = 8x^{II3} - 39x^{II2} + 24x^{II} - 4 = 0$	$x^{II} = 4 + \frac{1}{x^{III}}$
$X^{III} = 20x^{III3} - 96x^{III2} - 57x^{III} - 8 = 0$	$x^{III} = 5 + \frac{1}{x^{IV}}$
$X^{IV} = 193x^{IV3} - 483x^{IV2} - 204x^{IV} - 20 = 0$	$x^{IV} = 3 - \frac{1}{x^V}$
&c.	

Approximating values : 3, $\frac{10}{3}$, $\frac{37}{11}$, $\frac{195}{58}$, $\frac{622}{185}$, &c.
The roots : 3,36216.....

5. $x^3-12x-28=0$

.....

$X = x^3-12x-28=0$	$x = 4+\frac{1}{x^I}$
$X^I = 12x^{I3}-36x^{I2}-12x^I-1=0$	$x^I = 3+\frac{1}{x^{II}}$
$X^{II} = 37x^{II3}-96x^{II2}-72x^{II}-12=0$	$x^{II} = 3+\frac{1}{x^{III}}$
$X^{III} = 93x^{III3}-351x^{III2}-237x^{III}-37=0$	$x^{III} = 4+\frac{1}{x^{IV}}$
$X^{IV} = 649x^{IV3}-1419x^{IV2}-765x^{IV}-93=0$	$x^{IV} = 3-\frac{1}{x^V}$
&c.	

Approximating values : 4, $\frac{13}{3}$, $\frac{43}{10}$, $\frac{185}{43}$, $\frac{598}{139}$, &c.

The roots : 4·30213.....

In this method many advantages and abbreviations may be used. It may also be combined with the first method, when the approximation is somewhat advanced.

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(2) EQUATIONS WITH SEVERAL UNKNOWN MAGNITUDES.

Let $X=0$, $X_1=0$ be two equations involving x and y; and let it be assumed that we know already pretty near values of x and y; required to find nearer approximations to them.

I. Let $x=a$, $y=b$ be these values, and put for x, y in the equations $a+h$, $b+k$, retaining in the result

neither the product of h, k, nor their powers superior to the first. Then these results will be of the forms

$$A + Bh + Ck = 0$$
$$A' + B'h + C'k = 0$$

wherein A, B, C, A', B', C', are known quantities. Hence obtaining h and k, we get nearer approximations of x, y, viz. $a+h$ and $b+k$.

II. We proceed with these values exactly as we before did with a, b, and thus obtain new values of h, k, and nearer approximations to x, y.

III. This process being repeated will give the approximations to any required degree of accuracy. Moreover, in these calculations, logarithms may be used with advantage.

EXAMPLE.

Let the given equations be

$$x^7 - 5x^2y^4 + 1506 = 0$$
$$y^5 - 3x^4y - 103 = 0$$

The values $x = 2$, $y = 3$ nearly satisfy them.

First Correction.

$$14 - 1172h - 2160k = 0$$
$$-4 - 288h + 357k = 0$$

Hence $h = -0{\cdot}0035$, $k = +0{\cdot}0084$

Therefore $x = 1{\cdot}9965$, $y = 3{\cdot}0084$

Second Correction.

$$-0{\cdot}486-1189{\cdot}170h-2170{\cdot}576k = 0$$
$$0{\cdot}026-\ 287{\cdot}293h+\ 361{\cdot}890k = 0$$

Hence $h = -0{\cdot}000113$, $k = -0{\cdot}000161$

Therefore $x = 1{\cdot}996387$, $y = 3{\cdot}008239$

The last approximations to x, y, are right to five decimal places.

This method admits of being generalized thus. Let there be n equations $X = 0$, $X_I = 0$, $X_{II} = 0$, &c. involving the n unknown quantities x, y, z, &c. Then if a, b, c be nearer to their true values than by 1, assume $x = a+h$, $y = b+k$, &c. substitute in the given equations, and omit terms involving h^2, k^2, hk, &c. and we obtain n results of the form

$$A+Bh+Ck+ \text{ &c. } = 0,$$

from which h, k &c may be found; and therefore the second approximate values of x, y, z, &c., viz. $a+h$, $b+k$, &c.

With these values of x, y, z, &c. we proceed as before to find new values of h, k, &c. and so on to any degree of required accuracy.

SECTION III.

CONTAINING

PROBLEMS FOR APPLYING AND PRACTISING THE PRECEDING METHODS.

SIMPLE EQUATIONS.

XV. PROBLEMS FOR THE EQUATIONS OF THE FIRST DEGREE WITH ONE UNKNOWN QUANTITY.

1. TWO capitalists calculate their fortunes: it appears that the one is twice as rich as the other, and that together they possess 38700 crowns. What is the capital of each?

Ans. The one has 12900, the other 25800 crowns.

2. The sum of 2500 crowns is to be divided between two brothers, so that the one receives 4 crowns as often as the other receives one. How much does each receive?

Ans. The one receives 500 crowns, the other 2000 crowns.

3. A person has 2640 coins of two kinds, and $4\frac{1}{2}$ times as many of one sort as the other. How many has he of each sort? *Ans.* 480 and 2160.

4. Divide the number 237 into 2 such parts, that one may be contained in the other $1\frac{1}{4}$ times. What are these parts? *Ans.* $105\frac{1}{3}$, and $131\frac{2}{3}$.

What have these 4 problems in common?

5. To find two such numbers, that the one may be m times as great as the other, and that their sum $= a$ What are these numbers?

Ans. $\frac{a}{m+1}$, and $\frac{ma}{m+1}$.

6. A sum of 1200 crowns is to be so divided between 2 persons, A and *B*, *that* A's *share is to B's share* as 2 is to 7. How much does each receive?

Ans. *A* $266\frac{2}{3}$ crowns, *B* $933\frac{1}{3}$ crowns.

How may these problems be represented more generally?

7. Divide a number a into 2 such parts, that the 1st part : 2nd :: m : n. What are the parts?

Ans. $\frac{ma}{m+n}$, and $\frac{na}{m+n}$.

What has this problem in common with the 5th? And how may the one be reduced to the other?

8. How much money have I in my pocket, when the 4th and 5th parts of the same together amount to 11*s.* 3*d.*?

Ans. 1*l.* 5*s.*

9. Two friends met a horse-dealer leading a horse, which they resolved to buy jointly. When they had agreed as to the price, they found that the one was able to pay only the 5th, the other only the 7th part, this then

they put together, and paid the seller therewith in advance 48 crowns. How much did the horse cost?

Ans. 140 crowns.

This and the preceding problem are exactly similar. How may the similarity be represented by general expressions?

10. Find a number such, that when it is divided by m and n, and the quotients afterward added, the sum $= a$.

Ans. $\frac{mna}{m+n}$.

11. Divide the number 46 into two unequal parts, so that when the one is divided by 7, and the other by 3, the quotients together may amount to 10. What are these parts? *Ans.* 28 and 18.

12. Divide a number a into 2 such parts, that the sum of the quotients which are obtained when the one part is divided by m, the other by n, may be $=$ the number b. What are these parts?

Ans. $\frac{m(nb-a)}{n-m}$, and $\frac{n(mb-a)}{m-n}$.

13. In a company of 266 persons, consisting of officers, merchants, and students, we reckon four times as many merchants, and twice as many officers, as students. How many are there of each class?

Ans. 38 students, 152 merchants, and 76 officers.

14. A fortress has a garrison of 2600 men; among whom there are 9 times as many foot soldiers, and 3 times as many artillery soldiers as cavalry. How many then of each corps are there?

Ans. 200 cavalry, 600 artillery, and 1800 foot.

15. All my journeys taken together says a traveller, amount to 3040 miles; of which I have travelled $3\frac{1}{2}$ times as much by water as on horseback, and $2\frac{1}{3}$ times as much on foot as by water. How many miles did this man travel by each of the three ways?

Ans. 240 miles on horseback, 840 miles by water, and 1960 miles on foot.

What have the problems 13, 14, 15 in common?

16. Divide the number a into 3 such parts, that the 2nd may be m times, and the 3rd n times, as great as the first. What are these parts?

$$Ans. \quad \frac{a}{1+m+n}, \frac{ma}{1+m+n}, \frac{na}{1+m+n}.$$

17. I multiplied a certain number by 4, and divided the product by 3, when I obtained 24. What is the number? *Ans.* 18.

18. A field of 864 square rods is to be divided between 3 farmers A, B, C, so that A's part : B's :: 5 : 11, and C may receive as much as A and B together. How much does each receive?

Ans. A 135, B 297, C 432 square rods.

19. 1170 crowns are to be divided among 3 persons A, B, C, proportionally to their ages. Now B is a third part older than A, and C twice as old as A. How much does each receive? *Ans.* A 270, B 360, C 540 crowns.

20. On an approaching war 3 towns A, B, C, are to furnish their complement of 594 men; the division is to be made in proportion to their population. Now the population of A is to that of B, as 3 is to 5; the population of

B is to that of C, as 8 : 7 : . How many men must each town furnish? *Ans.* A 144, B 240, C 210 men.

21. A debt of 21000 crowns is to be divided between 4 creditors, A, B, C, D, in proportion to their claims. Now A's claim : B's :: 2 : 3; B's claim : C's :: 4 : 5; and C's claim : D's : : 6 : 7. How much then does each creditor receive?

Ans. A 3200, B 4800, C 6000, D 7000 crowns.

22. Divide the number a into 3 such parts, that the 1st : 2nd : : m : n; and the 2nd part : 3rd : : p : q. What are these parts?

Ans. $\frac{mpa}{mp+np+nq}, \frac{npa}{mp+np+nq}, \frac{nqa}{mp+np+nq}$.

23. The 3rd part of my income, says a person, I expend in board and lodging, the 8th part in clothes and washing, the 10th part in incidental expenses, and yet I save 318*l.* yearly. What is the amount of his yearly income? *Ans.* 720*l.*

24. A merchant finds by a fortunate speculation with his floating capital, he has gained 15 per cent, and that by these means it has increased to 15571*l.* What was his capital? *Ans.* 13540*l.*

25. A capital was put out for one year at 4½ per cent per annum; at the expiration of the year there was received back 13167*l.* capital and interest. How much did the capital amount to? *Ans.* 12600*l.*

26. The rent of an estate is greater by 8 per cent this than it was last year. This year's rent is 1890*l.* What was the former year's? *Ans.* 1750*l.*

27. A pound of a certain commodity is sold for 3*s.* 4*d.*: out of this there accrues to the seller a profit of $12\frac{1}{2}$ per cent. How much did it cost per cent?

Ans. 16*l.* 11*s.* 10*d.*

28. What capital is it, which, with 5 years' interest (the interest calculated at 4 per cent), amounts to 8208?

Ans. 6840*l.*

29. A gamester lost in the first game the 6th part, and in the second, the 10th part of the money he had about him, but won in the third game the 3rd part of it back. He counted his money, and found that he had won 3*l.* How much had he about him at first? *Ans.* 45*l.*

30. There are two numbers, whose sum is 96, and of which one is greater than the other by 16. What are the numbers? *Ans.* 40 and 56.

31. Two merchants having left off business, their profits amounting to 1200*l.* are to be divided between them in such a way, that one partner receives only half as much as the other, exclusive, however, of 50*l.* for the trouble. How much does each receive?

Ans. The one $766\frac{2}{3}l.$, the other $433\frac{1}{3}l.$

32. 1520*l.* are to be divided amongst 3 persons, A, B, C, so that B may receive 100*l.* more than A; but C 270*l.* more than B. How much does each receive?

Ans. A 350*l.* B 450*l.* C 720*l.*

33. A widow, according to the will of her deceased husband, wishes to divide a sum of 7500*l.* amongst her 2 sons and 3 daughters, so that each son may receive twice as much as each daughter; she herself, however, took exactly

as much as her children together, and yet there remained 500*l.* How much did the widow and each of her children receive?

Ans. The widow 4000*l.*, each son 1000*l.*, and each daughter 500*l.*

34. A company of 90 persons, consists of men, women, and children; the men are 4 in number more than the women, the children 10 more than the adults. How many men, women, and children are there in the company? *Ans.* 22 men, 18 women, and 50 children.

35. A forest of 8000 square feet is to be so divided amongst 3 farm-houses A, B, C, that B may receive 276 square feet less than A, but C 1112 square feet more than B. How much will each receive?

Ans. A 2480, B 2204, C 3316 square feet.

36. A father gives to his 5 sons 1000*l.* which are to be divided amongst them according to their ages, so that each 4 of the elder may receive 20*l.* more than his next younger brother. How much will the youngest receive? *Ans.* 160*l.*

37. A certain sum is divided amongst 3 persons A, B, C, as follows: A receives 3000*l.* less than the half, B 1000*l.* less than the third part, but C 800*l.* more than the fourth part of this sum. What is the sum to be divided? And how much does each receive?

Ans. The whole sum is 38400*l.*; A receives 16200*l.*, B 11800*l.*, C 10400*l.*

38. A dying man bequeaths in his will the half of his property to his wife, the 6th part to each of his 2 sons, the

12th part to his faithful servants, and the remaining 600*l.* to the poor. What was the amount of the property bequeathed? *Ans.* 7200*l.*

39. A meadow of 2850 square feet is to be divided amongst 3 land-lords A, B, C; A's share : B's :: 6 : 11, and C is to have 300 square feet more than A and B together. How much does each receive?

Ans. A 450, B 825, C 1575 square-feet.

40. A father leaves behind him 4 sons A, B, C, D, and a *fortune* of 2520*l.*, which they are to divide amongst them as follows: C is to receive 360*l.*, B as much as C and D together, but A twice as much as B diminished by 1000*l.* How much does A, B, and D receive? *Ans.* A 760*l.*, B 880*l.*, D 520*l.*

41. Five heirs are to divide an inheritance of 5600*l.* amongst them; B receives twice as much as A+200*l.* C three times as much as A, diminished by 400*l.*; D the half of what B and C receive together, and 150*l.* more; but D the fourth part of what his 4 participators have received, together with 475*l.* How much does each receive?

Ans. A 500*l.*, B 1200*l.*, C 1100*l.*, D 1300*l.*, E 1500*l.*

42. Five gamesters have lost jointly 40*l.* 15*s.*; B's loss amounts to 10*s.* more than treble A's; C's loss is 2*s.* less than twice B's; D lost 5*s.* less than A and B together, and E twice as much as B, diminished by 3*s.* How much did A lose?

Ans. A lost 1*l.* 17*s.* 5*d.*

43. A certain part of a commodity which weighed 40 cwt. was sold, and there remained 8 cwt. more than was sold. How many cwt. were sold? *Ans.* 16.

44. I once had 42*l.* in my possession; of this I gave away a certain sum, and there remained still three times as much as I had given away. How much did I give away? *Ans.* 10*l.* 10*s.*

45. Two gentlemen, A and B, play at billiards; A, before he began to play, had 42*l.*, and B 24*l.* After a match, in which each lost and won, A finds himself in possession of 5 times as much as B had remaining. How much did A win? *Ans.* 13*l.*

46. The garrison of a certain town consists of 1250 men, partly infantry and partly cavalry. Each horse-soldier receives 5*l.* monthly, and each foot-soldier 3*l.* If the monthly pay of the garrison amount to 4150*l.* how many horse and how many foot-soldiers did it contain? *Ans.* 200 cavalry and 1050 infantry.

47. A mason, 12 journey-men, and 4 assistants, received together 61*l.* wages for a certain time; the mason received 10*s.* daily, each journey-man 5*s.*, and each assistant 3*s.* How many days must they have worked for this money? *Ans.* 14·878 days.

48. A capitalist derives from the sums he has out at interest a yearly income of 2940*l.*; 4 fifths of it bears 4 per cent, and one-fifth 5 per cent. How much money has he out at interest? *Ans.* 70000*l.*

49. I have a certain number in my thoughts, says A to B, try to guess it. I multiply it by 7, add 3 to

the product, divide this by 2, subtract 4 from the quotient and obtain 15. Now what number is it? *Ans.* 5.

50. Find three such numbers, that the 2nd divided by the 1st, gives 2 for a quotient and 1 for a remainder; on the contrary, the 3rd divided by the 2nd gives 3 for the quotient with the remainder 3; the sum of these 3 numbers is 70. What numbers are they?

Ans. 7, 15, and 48.

51. How much money have you about you, said a person to his friend? I have so many shillings about me, said the other, that if I multiply their number by 5, subtract 3 from the product, multiply again the remainder by 4, and add 2 to the product, then take away from this number the cipher to the right, I obtain 23. How many shillings had he about him? *Ans.* 12.

52. An arithmetician desires his scholars to find a number, which he has in his mind, from the following data. If, says he, you multiply the number by 5, subtract 24 from the product, divide the remainder by 6, and add 13 to the quotient, you will obtain this same number. What number then is it? *Ans.* 54.

53. A boat which had started from a certain place 10 days, is pursued by another boat from the same place and by the same way. The 1st boat goes 4 miles every day, the other 9: how many days will the second take to catch the first? *Ans.* 8 days.

54. A boat left this place n days ago, which goes a miles daily; this is pursued by another which goes b

miles daily: how many days will the 2nd require to overtake the first?

Ans. $\frac{na}{b-a}$ days.

55. But in what time will the 2nd boat overtake the 1st, when it is supposed that the 2nd starts 12 days later than the first; and that its speed is to that of the 1st, as 8 to 3? *Ans.* In $7\frac{1}{5}$ days.

56. Two bodies start from the same place one after the other in a straight line; the 2nd starts n seconds later than the 1st, and its speed is to the first's as q is to p. In what time will these bodies be together?

Ans. $\frac{pn}{q-p}$ seconds after the setting-out of the 2nd.

57. A courier, who goes $31\frac{1}{2}$ miles every 5 hours, is sent from a certain place; when he had been gone 8 hours another was sent after him, and this one, in order to overtake the 1st, must go $22\frac{1}{2}$ miles every 3 hours. When will the 2nd overtake the 1st?

Ans. 42 hours after the departure of the 2nd courier.

58. When all the conditions in the preceding problem remain the same, except that the 1st courier, besides the advantage of starting earlier, has this one also, that he travelled from a place 36 miles farther on the road: in this case in how many hours will they come together?

Ans. In 72 hours after the departure of the 2nd Courier.

59. In order to give the preceding problem the requisite generality, let the place from which the 1st starts, be situated a miles in advance on the road; further,

let the number of the hours, by which he had the start of the other be $= b$; let the speed of the 1st courier be such, that he goes c miles in d hours, and the speed of the 2nd be so great that he goes e miles in f hours. In how many hours after the departure of the 2nd courier will they be together?

Ans. In $\frac{(ad + bc)f}{de - cf}$ hours.

60. But in how many hours will they come together, when the 1st courier, instead of starting from a place a miles in advance, starts from one as many miles backwards. What must be done in order to adapt the solution of the preceding problem to this case?

Ans. In $\frac{(bc - ad)f}{de - cf}$.

61. A regiment marches from the place A in a direct line to the place B, and goes $3\frac{1}{2}$ leagues each day; eight days after, another regiment marches from the place B in a straight direction to A, and goes $5\frac{1}{8}$ leagues each day. If both places be 80 leagues distant from one another, on what day after the departure of the 1st will these regiments meet? *Ans.* On the 14th day.

What values must we assign to the quantities a, b, c, d, e, f, in the 59th problem, in order to adapt the solution there given to the present case, if we did not wish to solve this problem anew?

62. Two bodies move in opposite directions, one runs c feet in each second, the other C feet. The two places, from which they start at the same time, are distant d feet from one another. When will they meet?

Ans. In $\frac{d}{C + c}$ seconds.

63. But in what time will these 2 bodies come together, when that which goes C feet each second, runs after the other?

Ans. $\frac{d}{C-c}$ seconds.

Is the problem, as it is here stated, always possible? And what is required for it to be possible?—What does the expression $\frac{d}{C-c}$ signify, when $C = c$? What does it denote when $C < c$?

64. A hostile corps has set out two days ago from a certain place, and goes 27 miles daily. Another corps wishes to go in pursuit of it from the same place, and so quickly, that it may reach the other in 6 days. How many miles must it go daily to accomplish it?

Ans. 36 miles.

65. Two bodies move after one another in the same direction, the 1st has a start of d units of measure (for instance, feet), and of t units of time (for instance, seconds); the 1st, in each unit of time runs over c units of measure. How many units of time will the 2nd require to overtake the 1st?

Ans. $\frac{d+ct}{C-c}$ units of time.

66. On the contrary, how many units of time will the 2nd take, when the bodies, instead of running after one another, run towards one another, and all the other conditions remain the same.

Ans. $\frac{d-ct}{C+c}$ units of time.

How may this expression be derived from that found in the preceding problem?

67. At twelve o'clock both the hands of a clock are together. When, and how often will these hands be together in the next 12 hours?

Ans. The hands will meet 11 times; these rencontres will be at $5\frac{5}{11}$ minutes past one, $10\frac{10}{11}$ minutes after 2, $16\frac{4}{11}$ minutes after 3, and so on; viz. in each successive hour $5\frac{5}{11}$ minutes later.

68. Two bodies move after one another in the periphery of a $\odot$, which measures p feet. At first they are distant from one another by an arc measuring d feet; the 1st moves c feet, the 2nd C feet in a second. When will these two bodies meet for the 1st time, 2nd time, and so on, supposing that they do not impede each other's motion?

Ans. In $\frac{d}{C-c}$, $\frac{p+d}{C-c}$, $\frac{2p+d}{C-c}$ &c. seconds.

69. But when will they meet, when the 1st begins to move t seconds sooner than the 2nd?

Ans. In $\frac{d+ct}{C-c}$, $\frac{p+d+ct}{C-c}$, $\frac{2p+d+ct}{C-c}$ &c. seconds.

70. But if the 1st start t seconds later than the 2nd, when will they meet?

Ans. In $\frac{d-ct}{C-c}$, $\frac{p+d-ct}{C-c}$, $\frac{2p+d-ct}{C-c}$ &c. seconds.

71. But if the 1st, instead of preceding the 2nd, starts from the same place t seconds sooner, when do they meet?

Ans. In $\frac{d-ct}{C+c}$, $\frac{p+d-ct}{C+c}$, $\frac{2p+d-ct}{C+c}$ &c. seconds.

72. But when they do meet again, if the 1st starts

from the same place as the 2nd does, but with this difference, that it begins to move t seconds later?

Ans. In $\frac{d+ct}{C+c}$, $\frac{p+d+ct}{C+c}$, $\frac{2p+d+ct}{C+c}$ &c. seconds.

Whence arises the difference in the signs in the solutions of these 5 similar problems, since otherwise the expressions are in all respects alike? What do these expressions mean in the 68th, 69th, and 70th problems, when $C = c$? What do they indicate, when $C < c$?

What is a compound proportion? and where are examples to be found of it? The proportion of 2 quotients may be compounded of 2, 3, and more simple proportions. It is of the greatest importance in Algebra.

73. From two orifices of a reservoir, of different sizes, the water runs with unequal velocities. We know that the orifices are as 5 : 13, and that the velocities of the fluid are as 8 : 7; we know further, that in a certain time there issued from the one 561 cubic feet more than there did from the other. How much water, then, did each orifice discharge in this space of time?

Ans. The one 440, the other 1001 cubic feet.

74. A dog pursues a hare. Before the dog started, the hare had made 50 paces, and this is the distance between them at *first*. The hare takes 6 paces to the dog's 5, and 9 of the hare's paces are equal to 7 of the dog's. How many paces can the hare take before the dog overtakes her?

Ans. 700 paces.

75. Two bombardiers throw different kinds of shells from a battery. The 1st had already thrown 36 times, be-

fore the 2nd began to throw; and throws 8 times to the 2nd's 7; on the other hand, the 2nd uses as much powder for three throws as the 1st does for four. How many throws must the 2nd make in order to consume as much powder as the 1st? *Ans.* 189 throws.

76. How happens it, said one traveller to another, that you have out-run me 3000 feet, although I took twice as long steps as you? That I allow, the other answered him, but I, on the other hand, took 5 times as many steps as you. If, then, all this be correct, how many feet must each have travelled?

Ans. The 1st 2000, the 2nd 5000, feet.

77. In order to make the foregoing problem more general, let the number of feet, by which the 2nd pedestrian has out-run the 1st, $= a$; let the length of their steps be as $b : c$, and their number as $d : e$. What expressions give the distance they went over?

Ans. $\frac{abd}{ce-bd}, \frac{ace}{ce-bd}$.

78. Two quotients, whose difference $= d$, for some reason, are to one another as $m : n$; but for some other reason, respecting which it is assumed, that it does not hinder the operation of the other, they are as $m' : n'$. How are these quotients expressed?

Ans. $\frac{mm'd}{nn'-mm'}, \frac{nn'd}{nn'-mm'}$, are the expressions for these quotients.

What values must we assign to the letters m, m', n, n', d, that the problems 73, 74, 75, 76, 77, may come under these conditions?

79. Two quotients, whose difference $= d$, for certain reasons, of which it is assumed, that they have no mutual influence on one another, are as $m : n$, $m' : n'$, $m'' : n''$ How are these two quotients expressed?

Ans. $\frac{m\,m'\,m''\,d}{n\,n'\,n'' - m\,m'\,m''}$ and $\frac{n\,n'\,n''\,d}{n\,n'\,n'' - m\,m'\,m''}$

80. But when the sum s of these quotients is given instead of greater difference, how are they expressed?

Ans. By $\frac{m\,m'\,m''\,s}{n\,n'\,n'' + m\,m'\,m''}$ and $\frac{n\,n'\,n''\,s}{n\,n'\,n'' + m\,m'\,m''}$.

81. A person puts a capital of 5500*l.* out at interest at 4 per cent, and $4\frac{1}{2}$ years after another capital of 8000*l.* at 5 per cent. If he leave these two capitals constantly at interest; in how many years will he have drawn the same interest from both?

Ans. In 10 years from the time that he put out the 1st sum.

82. A person possesses a waggon with a mechanical contrivance, by which the difference of the number of revolutions of the wheels on a journey may be determined. It is known that each of the two fore-wheels is $5\frac{1}{4}$, and that each of the two hind-wheels is $7\frac{1}{3}$ feet in the circumference. When in a journey the fore-wheel has made 2000 revolutions more than the hind-wheel, how great was the distance travelled? *Ans.* 39900 feet.

83 When the fore-wheel of the above-mentioned vehicle is a feet, and the hind-wheel b feet in circumference, how great will be the distance gone over, when the

fore-wheel has made n revolutions more than the hind-one?

Ans. $\frac{abn}{b-a}$ feet.

84. A wine-merchant has two kinds of wine; the one costs 36 pence a quart, the other 20 pence. He wishes now to mix both wines together in such quantities, that he may have 50 quarts, and each quart without profit or loss, may be sold for 30 pence. How much must he take of each sort to make up this mixture?

Ans. $31\frac{1}{4}$ quarts of the best, and $18\frac{3}{4}$ quarts of the worst sort.

85. Let the price of the best wine in the preceding problem be $= a$ pence, the price of the worst $= b$, the number of quarts which are produced by the mixing $= n$, and the price of this mixture $= c$. How much must be taken of each sort?

Ans. $\frac{(a-c)n}{a-b}$ quarts of the worst, and $\frac{(c-b)n}{a-b}$ of the best sort.

86. A goldsmith has two kinds of silver, viz. one kind contains 14 parts of pure silver (∴ each pound of which contains 14 oz. of pure silver, and 2 oz. alloy), and the other 8. He cannot, however, use that which is so good as the 1st, nor that which is so bad as the 2nd; he wishes then to make a dish, which shall be 20lb in weight, each lb. of it containing 12oz. of silver. How many lbs. must he take of each kind, so that, by melting them together, he may obtain both the desired weight and the desired alloy?

Ans. $13\frac{1}{3}$lbs. of the best, $6\frac{2}{3}$lbs. of the worst silver.

87. A wine-merchant has 40 bottles of wine, each of which he sells for 7 shillings. Since, however, this price he deems too high for his customers, he wishes, as he intends to deal fairly, to add as much water to it, as will enable him to sell a bottle of the mixed wine for 4*s*. 6*d*. How much water must he add to it?

Ans. $22\frac{1}{2}$ bottles.

88. A person has 35lbs. of silver, each of which contains 15 out of 16 parts pure, and he wishes to add so much copper to it, that the lb. may contain only 12oz. of pure silver. How much copper must he add? *Ans.* $8\frac{3}{4}$lbs.

89. How much silver of 8 parts out of 16 pure must we melt with $7\frac{1}{2}$lbs. of 13 parts pure, that the contents may be brought to 9 parts pure? *Ans.* 30*lbs.*

90. A person wishes for 17 silver coins, consisting of crowns and shillings, for 2*l*. 13*s*. How many coins of each sort can he receive for them?

Ans. 9 crowns, and 8 shillings.

What has this problem in common with the 84th and 86th? And how may this common property be expressed by words?

91. Required two numbers whose sum $= a$, and which are of such a nature that, if the 1st be multiplied by m, the other by n, the sum of the products may be $= b$. What expressions will give these numbers?

Ans. $\frac{b-na}{m-n}, \frac{ma-b}{m-n}$.

What do these expressions mean, when $m=n$? What when at the same time, $b = na = ma$?

92. One of my acquaintances is now 40 years old, his son 9 years. In how many years will this man, who is now more than four times as old as his son, be only twice as old? *Ans.* In 22 years.

93. One of my acquaintances is now 30, his elder brother 20, and consequently 3 : 2 is the proportion of his age to that of his brother's. In how many years will the proportion be as 5 : 4? *Ans.* In 20 years.

94. But, on the other hand, how many years is it since he was 6 times as old as his brother?

Ans. 18 years.

95. He has, however, besides the above-mentioned brother, another, who is now only 6 years old. When will both his brothers be together as old as he is himself?

Ans. In 4 years.

96. His father is now 49 years old, and consequently the 3 brothers are together 7 years older than their father; there was, however, a time when the father was exactly as old as his 3 sons together. How long is that ago?

Ans. $3\frac{1}{2}$ years.

97. At one time his father said to him (the youngest son not being then born) that he was the 4th part older than both his sons together. How long is this ago?

Ans. 9 years.

98. Saltpetre and sulphur are mixed together in a mass of 80lbs., and in such a proportion, that for every 7 parts of saltpetre there are 3 parts of sulphur. How much saltpetre must still be added to the mass, so that the proportion of these substances may be such,

that for every 11 parts of saltpetre there may be 4 parts of sulphur? *Ans.* 10 lbs.

99. On the other hand, how much sulphur must be taken away from the mass, in order that the proportion of 11 : 4 may be obtained? *Ans.* $3\frac{7}{11}$ lbs.

100. But if there be as much saltpetre added, as there is taken away of the sulphur, so that the weight of the whole mass remain unchanged: how much saltpetre must be added to the mass, and in its stead taken away of the sulphur? *Ans.* $2\frac{2}{3}$ lbs.

101. In a company there were, at first, three times as many gentlemen as ladies; but afterwards, when 8 gentlemen with their wives went away, the proportion of the persons present of both sexes was still more unequal, for there then remained 5 times as many gentlemen as ladies. Of how many of each sex did this company originally consist? *Ans.* Of 48 gentlemen and 16 ladies.

102. What number must be added to each of the 2 given numbers a and b, that the sums may be as $m : n$? Or, which means the same, to what number must a and b be added, that the sums may have the given proportion $m : n$? *Ans.* $\frac{mb-na}{n-m}$ is the number sought.

103. What number must be added to a, and subtracted from b, that the sum may be to the difference as $m : n$?

Ans. $\frac{mb-na}{m+n}$.

104. What number must be substracted from a and b, that the differences may be as $m : n$? *Ans.* $\frac{na-mb}{n-m}$.

In these 3 last problems, the problems 92, 93, 94, 98, 99, 100, are contained as simple cases. What values must we assign to the quantities a, b, m, n, in order to adapt the forms to those cases?

105. In a full wine-cask there are 3 bung-holes, by the 1st the wine can be drawn off in 2, by the 2nd in 3, and by the 3rd in 4 hours. What time will it take to draw it off when all 3 bung-holes are spouting at once?

Ans. $55\frac{5}{13}$ minutes.

106. A cistern can be filled by 3 pipes; by the 1st in $1\frac{1}{3}$ hours, by the 2nd in $3\frac{1}{3}$ hours, and by the 3rd in 5 hours. In what time will this cistern be filled when all 3 pipes are opened at once?

Ans. In 48 minutes.

107. In order to make the foregoing problem more general, let the time in which the 1st pipe takes in filling alone the cistern $= a$, the time which the 2nd takes in doing it $= b$, and the time which the 3rd requires $= c$. What expression gives the time, in which all three pipes together will fill it?

Ans. $\dfrac{abc}{ab+ac+bc}$.

108. What expression will give the time, in which 4 pipes will fill the cistern, when they singly fill it in the times a, b, c, d, respectively?

Ans. $\dfrac{abcd}{abc+abd+acd+bcd}$.

109. Three masons are employed in building a wall. The 1st does 8 cubic feet in 5 days, the 2nd 9 cubic feet in 4 days, and the 3rd 10 cubic feet in 6 days. How

much time will it take these 3 masons, when they work together, to build 756 cubic-feet of the wall?

Ans. $137\frac{15}{331}$ days.

110. A mechanic can finish a certain work expressed by a, in a space of time expressed by b; another mechanic can finish the work c in the time d, and a 3rd the work e in the time f. In what time will these 3 mechanics together accomplish the work g?

Ans. In the time $\frac{bdfg}{adf+bcf+bde}$. The unit of time in this expression is that to which the quantities b, d, f refer.

111. A cistern of $755\frac{1}{4}$ cubic feet is filled by 3 pipes. The 1st discharges 12 cubic feet in $3\frac{1}{4}$ days, the 2nd $15\frac{1}{3}$ cubic feet in $2\frac{1}{2}$ days, and the 3rd 17 cubic feet in 3 days. In what time will the cistern be filled?

Ans. In $48\frac{3}{4}$ days.

112. Three causes respectively produce in the times t, t', t'', the effects e, e', e''. What time will be required when all 3 causes operate together, in order to produce the effect E, supposing that they act independently of one another?

Ans. The time is $\frac{Ett't''}{et't''+e'tt''+e''tt'}$. The unit of time the same as that in which the times t, t', t'' are given.

In this last the problems 105, 106, 107, 108, 109, 110, 111 are contained. How so?

113. A person has 3 pieces of metal of equal size. Of the 1st 5 cubic inches weigh $69\frac{3}{4}$ ounces; of the 2nd

$3\frac{1}{3}$ cubic inches weigh 41 ounces, and $4\frac{2}{7}$ cubic inches of the 3rd weigh 91 ounces. If then the 3 pieces together weigh $949\frac{2}{3}$ ounces, what is the size of each?

Ans. 20 cubic inches.

114. A charitable man wished to collect some money in a numerous company for a poor person. Each of the company offering to contribute 16*s.*; the collector considered it too much, because he would then collect 6*l.* more than he required for the man. He therefore proposed, that each should only give 10*s.*; but on calculation then found, that it would be too little, because he then would fall short of the required sum by 12*l.* How many persons did the company consist of? How much did the man need? And how much must each contribute, in order to collect the sum required?

Ans. The company consisted of 60 persons; the sum which was to be collected was 42*l.*; and the contribution of each was 14*s.*

115. A merchant, in order to pay a debt, is obliged to lower the price of a certain commodity to the prime cost. On account of bad book-keeping, he neither knows the weight nor the prime cost of the article. He only remembers this much, that if he had sold the lb. for 30*s.*, he would have gained 5*l.* by it, and if he had sold the lb. for 22*s.* he would have lost 15*l.* by it. What, according to these conditions, were the weight and the prime cost of the article?

Ans. The weight was 50 lbs., and the prime cost 26*s.* per pound.

116. A person wishes to raffle away a gold watch, and

for that purpose makes a certain number of tickets. If he sell the ticket for 5*s*. he would lose 5*l*., because the watch cost him more than he would in this case get; but if he sell the ticket for 6*s*. then he gains 4*l*. How much therefore did the watch cost him, and how many tickets did he sell? *Ans.* 50*l*., and 180 tickets.

117. A master-mason has engaged a number of masons for the erection of a building. He finds, after entering into a calculation, that if he gave each mason m shillings a day, he would daily expend 1*s*. less than were assigned for that purpose by the estimate; and that he would lose 6*s*. if he should give each n shillings a day. How many men did he hire? and how much was the fixed daily wages of each?

Ans. The number of masons was $\frac{a+b}{n-m}$, and the daily wages $\frac{an+bm}{n-m}$ shillings.

118. If a certain number be multiplied by m and m', we obtain 2 products, which exceed a certain other number by a and a'. What are the two numbers?

Ans. The first is $\frac{a-a'}{m-m'}$, the 2nd $\frac{m'a-ma'}{m-m'}$.

What values must be assigned to the letters m, m', a, a', that the problems 114, 115, 116, 117, may be comprehended under this?

119. Find a number which has this property, that if it be multiplied by 5 it gives a product which is as much above 20 as the number itself is under 20. What number is it?

120. In order to meet all my expenses, said a person, I ought to have a yearly income of 540*l.*; but it wants a considerable sum of this. Were my income $3\frac{1}{2}$ as great as it really is, then I should be able not only to meet my expenses, but even to save as much above my income as I am now deficient. What was this man's yearly income?

Ans. 240*l.*

121. A transcriber was asked how many sheets he wrote weekly; he answered, "I only work 4 hours a day, and cannot finish 70 sheets, which I could wish to do; but if I could work 10 hours a day, then I should write weekly exactly as many above 70 sheets as I now write less than this number." How many sheets did he write weekly? *Ans.* 40.

122. A person asked a surveyor, how far the two signals, whose distance he had just measured, were from one another. He answered, "Their distance is far short of 1000 feet; for if I add to this distance its 3rd part, increase this sum by 176, and multiply this last by $2\frac{1}{2}$, then I shall as much exceed 1000, as the distance falls short of it." How far were the signals from one another?

Ans. 360 feet.

123. A person wished to buy a house, and in order to raise the requisite capital, he draws the same sum from each of his debtors. He tried whether, if he obtained 250*l.* from each, it would be sufficient for the purpose; he found, however, that he would then have 2000*l.* too little. He tried it, therefore, with 340*l.*, but this gave him 880*l.* more than he required. How many debtors

had he? What was the sum to be raised? And how much must he exact from each of his debtors?

Ans. The number of his debtors is 32, the capital 10000*l.*, and the sum which he exacted from each 312*l.* 10*s.*

124. A merchant is to make the following payments at 3 different periods; 2832*l.* in 3 months, 2560*l.* in 9 months, and 1450*l.* in 16 months. The creditor wishes to receive the whole sum of 6842*l.* at once. When ought the payment to be made? *Ans.* In 8 months.

125. A person has to make the following payments at 4 different periods: one sum a in l months, a sum b in m, a sum c in n, and a sum d in p months. If he wish to pay his whole debt $a+b+c+d$ at once, at what period must he do it?

Ans. In $\frac{al+bm+cn+dp}{a+b+c+d}$ months.

126. A capitalist engages to lend a merchant 16000*l.* for 15 months; but as he is not able to furnish this sum at once, they agree to this, that the capitalist shall 1st furnish 5000*l.*, after the expiration of 6 months 3000*l.* more, and then again in 8 months the last 8000*l.* How much longer can the merchant retain the same capital of 16000*l.*, so that neither party may suffer?

Ans. $9\frac{1}{2}$ months.

127. A land proprietor entered into a contract with his neighbour, in which he bound himself to take 400 of his neighbour's oxen in to pasture. The neighbour, however, sent, with the land-owner's consent, at first only 200 head; in 7 months 250 more; and 8 months after that

again, 150 more. How much longer must the Grazier feed all the 600 oxen, so as to fulfil the engagement he has entered into? *Ans.* $2\frac{1}{3}$ months.

129. A person purchases a certain commodity for 4500*l.*, which he is to pay for at the expiration of a year. He agrees with the seller to pay him 1500*l.* cash, and the remaining 3000*l.* at 4 equal periods, 750*l.* each time. What period must be fixed upon so that neither party may be a loser? *Ans.* A period of $7\frac{1}{2}$ months.

130. A certain sum is to be paid as follows: 1376*l.* in 5 months; 3 months after this 2560*l.*, and the remainder 5 months after this. If the whole sum be paid at once, it must take place in 10 months. What was the whole sum to be paid? *Ans.* 7936*l.*

131. A person is to pay 7000*l.*; viz. 2000*l.* in $3\frac{1}{2}$ months, 3500*l.* in 4 months, and 1500*l.* in 14 months. His creditors propose to him to pay this sum in 2 different instalments, the half each time, so that the 2nd payment may be a month later than the 1st. If the debtor agree to this, in what time must the 1st instalment be paid?

Ans. In $3\frac{2}{3}$ months.

132. Two persons bid for a garden which is to be sold. The one offers 7705*l.*; viz. 3365*l.* cash, and 4340*l.* in 8 years; or, if the seller wish it, he will give this last sum also in cash, deducting 3 per cent discount. The 2nd bids another sum, which, calculated from this time, is to be paid in 3 equal instalments, each at an interval of 2 years; viz. the 1st third part in 2 years, the 2nd in 4 years, and the last third in 6 years; or, if the seller wish

it, cash immediately, deducting 5 per cent discount. The seller considers both offers as the same. How much did the 2nd bid? *Ans.* 7722*l.*

133. Three merchants, A, B, C, enter into partnership. A advances 1200*l.*, B 800*l.*, and C 600*l.* A leaves his money 8 months, B 10 months, and C 14 months in the business. They gain 500*l.* How much does each receive of this sum?

Ans. A receives $184\frac{8}{15}l.$, B $153\frac{11}{15}l.$, C $161\frac{7}{15}l.$

134. Let A's capital in the preceding problem be equal to a, B's $= b$, C's $= c$; A leaves his money l months, B m months, and C n months in the business. Let the profit $= g$. What share does each receive of the profit?

Ans. A's share is $= \frac{alg}{al + bm + cn}$, B's $= \frac{bmg}{al + bm + cn}$

and C's $= \frac{cng}{al + bm + cn}$.

135. Three merchants entered into trade. The 1st contributed 17000*l.*, the 2nd 13000*l.*, the 3rd 10000*l.* As they must have some person to conduct the business, the one who had furnished the least, offered to undertake the management of it; but under this condition, that he receives 3 per cent profit, besides what he is entitled to from his deposited capital. Now it is found that they have gained $3526\frac{1}{2}l$, How much is due to each?

Ans. To the 1st 14875*l.*, to the 2nd 11375*l.*, to the 3rd $9012\frac{1}{2}l.$

136. Three creditors claim an estate, which, after deducting the proper expenses, amounts to 3139*l.*; the claim

of the 1st was 2000*l.*, the 2nd's 2500*l.*, and that of the 3rd 3500*l.* But as the estate is not sufficient to satisfy all these creditors, and, moreover, their claims were not equally legal, the sum, by a judicial decree, is to be divided amongst the creditors in proportion to their demands, yet for the alleged reason, the 2nd receives 10, and the 3rd 25 per cent more than his share. How much, therefore, does each receive?

Ans. The 1st 688*l.*, the 2nd 946*l.*, the 3rd 1505*l.*

137. Let the estate in the preceding problem be $= a$, the claims of the 3 creditors f, g, h; let the 2nd receive m per cent, and the 3rd n per cent more than the 1st. How much does each receive?

Ans. The 1st $\dfrac{100af}{100(f+g+h)+gm+hn}$

The 2nd $\dfrac{100(m+)ag}{100(f+g+h)+gm+hn}$

The 3rd $\dfrac{(100+n)ah}{100(f+g+h)+gm+hn}$.

How must these formula be altered, if the 2nd, instead of receiving m per cent more, should get m per cent less?

138. Three persons, A, B, C, contribute a certain sum for business; B contributes $\frac{1}{2}$ more than A, and C 300*l.* more than A and B together. After some time, the profit amounting to 5020*l.*, is divided, and C receives 2570*l.* for his share. How much did each contribute?

Ans. A 2450*l.*, B 3675*l.*, C 6425*l.*

139. Three merchants, A, B, C, formed themselves into a trading company; C contributed 5600*l.*, but A 320*l.* less than B; A leaves his money 7 months, B 14

months, and C 12 months in the business. The profits, amounting to 2402½*l.*, are divided amongst the partners in proportion to the sum advanced, and the time; now B receives 876⅔*l.* for his share. How much did A and B advance? *Ans.* A 3450*l.*, B 3770*l.*

140. A person dies and leaves 4 sons, and a fortune of 1100*l.* Ten months after his death the will was first opened, and in this time the children had expended all the property they had inherited, together with the interest. At the same rate of expenditure, and the same rate of interest, 3 other children spent a capital of 1200*l.* in 15 months. How high were the interests of these sums calculated? And how long would it take 6 children, under the same circumstances, to expend 1650*l.*?

Ans. The interests amounted monthly to ⅔ per cent, and it takes 10 months to spend the 1650*l.*

141. Five brothers in the space of 9 months ran through a capital of 4800*l.*, together with the interest for this whole time. At the same rate of expenditure, 2 other persons in 16 months ran through a capital of 3320*l.*, together with the interest. The rate of interest was in both cases the same. How much did each spend monthly?

Ans. 110⅔*l.*

142. A servant received from his master 40*l.* wages yearly, and a suit of livery. After he had served 5 months, he asked for his discharge, and received for this time the livery, together with 6⅛*l.* in money. What did the livery cost? *Ans.* 18*l.*

143. A builder employs 2 journeymen, who work for

him at the same rate. To one he gave 4 bushels of rye and 14*l.* for 56 days' work; and to the other $7\frac{1}{2}$ bushels of rye and 17*l.* 5*s.* for 84 days labour. What did he calculate the bushel of rye at? *Ans.* 2*l.* 10*s.*

144. A master hired a journey-man, and promised him 6*s.* 8*d.* each day that he worked for him; but if he work any where else, then the journey-man must pay him 4*s.* 2*d.* for his board. At the expiration of 50 days they settle, and the journey-man receives 9*l.* 1*s.* 8*d.* How many days has he worked for his master? *Ans.* 36.

145. A peasant brought a basket of eggs to market and offered them at 7 farthings a-piece. A person who was passing by stumbled against the basket and broke 5 of his eggs. Being paid for them, he resolved to sell the remainder at 8 farthings a-piece, because he would then get as much by them as he would have received before by his full number, besides the money for the eggs that were broken. How many eggs did the peasant bring to market? *Ans.* 40.

146. A cook, who was carrying lemons, was asked how many he had. As he was a good arithmetician, he answered in the following enigmatical way: "These lemons cost me 1*s.* 6*d.* the dozen; but if I had got the 5 into the bargain which I asked for, then they would have cost me $2\frac{1}{2}$*d.* a dozen less." How many lemons had he?

Ans. 31.

147. A merchant sends for a piece of cloth, for which he pays 2*l.* 10*s.* per ell. In measuring it, however, he finds, that it contains, in fact, 5 ells more than were

charged him, but, at the same time, of so bad a quality, that, for want of opportunity to send it back, he is obliged to sell it at 2*l.* per ell. He calculates, that he can do this by losing merely $13\frac{1}{3}$ per cent. How many ells did the piece contain?

Ans. By statement 60, but in fact 65.

148. Now, said a person, I spend the 7th part of my income at the theatre, and the rest in my ordinary expenses; could I, however, receive an addition to my income of 100*l.* then I might spend the 5th part of it at the theatre, and still have 40*l.* more than before for my ordinary expenses. What was his income? *Ans.* 700.

149. An admirer of the sciences, who, till now, had expended the 4th part of his yearly income in the purchase of books, resolves, on account of an addition which he had received to it, to lay out in future the 3rd part of it in this way, because he calculates, that, notwithstanding this increase, he will still have as much left for his ordinary expenses as before. What proportion did this addition bear to his former income?

Ans. It amounted to the 8th part.

150. In a certain town, each house-keeper formerly contributed the 7th part of his house-rent for taxes, afterwards the tax was raised, and each was obliged to pay the 6th part. How much must he raise his lodgers, in order that he may receive as much after paying the tax as he did before? *Ans.* The 35th part of the former rent.

151. I once had an untold sum of money lying before me. From this, I first took away the 3rd part, and

put in its stead 50*l.* A short time after I took from the sum thus augmented, the 4th part, and put again in its stead 70*l.* I then counted my money and found 120*l.* What was the original sum? *Ans.* 25*l.*

152. From a sum of money, 50*l.* more than the half is first taken away; from the remainder 30*l.* more than its 5th part; and again, from the 2nd remainder, 20*l.* more than its fourth part. At last there remained only 10*l.* What was the original sum? *Ans.* 275*l.*

153. A rich man bequeathed in his will a certain sum, which was to be divided amongst three of his domestics as follows: The valet's share is 200*l.* together with the half of the remainder; the cook's the 5th part of the residue and 400*l.*; the remainder, which amounts to 520*l.*, the coachman receives. How much did the legacy amount to? *Ans.* 2500*l.*

154. A country-man brings his eggs to market, and first sells 4 more than the half of them; then he goes farther, and sells the half of the remainder and 2 over. A wag runs off with 6 more than the half of what were left, and he then goes home with 2 eggs remaining in his basket. How many eggs did the peasant take to the town? *Ans.* 80.

155. A merchant adds yearly to his capital one third, but takes from it, at the end of each year, 1000*l.* for his expenses. At the end of the 3rd year, after deducting the last 1000*l.*, he finds himself in possession of twice the sum he had at first. How much did he possess originally? *Ans.* 11100*l.*

156. A merchant increases his capital yearly by 20

per cent, but takes from it every year 1000*l.* for the support of himself and family. After he had carried on his business in this manner for 3 years, he finds, after deducting the usual 1000*l.*, that his capital has increased 200*l.* more than three-fifths of the original sum. What was this capital? *Ans.* 30000*l.*

157. A man, to please his children, brings home a number of apples, and divides them as follows: To the 1st and eldest of them he gives the half of his whole stock, diminished by 8; to the 2nd the half of the remainder, diminished by 8; and he does the same with the 3rd and 4th. After this he gives the 20 remaining apples to the 5th. How many apples did he bring home? *Ans.* 80.

158. I take a certain number, multiplying it by $3\frac{3}{7}$, take 60 from the product, multiply the remainder by $2\frac{1}{2}$, and after this subtract 30, when nothing remains. What is the number? *Ans.* 21.

159. A spend-thrift lends his fortune at 4 per cent interest. After he had let it remain 2 years, he took out the 4th part of it, and allowed the remainder to stand 7 months. At the expiration of this time he took, once more, the 4th of the remainder, and allowed the capital thus diminished to stand 13 months, when he demanded all his remaining fortune. In the space of 44 months he had drawn no less than $6093\frac{3}{4}$*l.* of interest money. What was his fortune at first? *Ans.* 50000*l.*

160. A father leaves a number of children and a certain sum which they are to divide amongst them as

follows: The 1st is to receive 100*l.* and then the 10th part of the remainder; after this the 2nd has 200*l.* and the 10th part of the residue; again, the 3rd receives 300*l.* and the 10th part of the remainder, and generally, each succeeding one receives 100*l.* more than the one immediately preceding, and then the 10th part of that which still remains. At last it is found, that all the children have received the same. What was the fortune left? And how many children were there alive?

Ans. The fortune was 8100*l.*, and the number of children 9.

160. But what must the sum left, and the number of children be, when the 1st child receives 30*l.* and the 9th part of the remainder; the 2nd 60*l.* together with the 9th part of the remainder; and, generally, each succeeding one 30*l.* more than the one immediately preceding, together with the 9th part of the residue, and yet all have the same sum? *Ans.* 1920*l.*, and 8 children.

161. Farther, what must the fortune and number of children be, when, in general, the 1st receives £*a.* together with the *n*th part of the remainder; but each succeeding child £*a.* more, together with the *n*th part of the remainder, and it is found at last, that they have all received the same?

Ans. The fortune $= (n-1)^2 a$, the number of children $= n - 1$.

162. A general wishing to draw up his regiment in square, tried it in two ways. The 1st time he had 39 men over; the 2nd, having extended the side of the square by

1 man, he wanted 50 men in order to compleat the square. How strong was the regiment?

Ans. 1975 men.

163. A person has a certain number of sovereigns which he wishes to arrange in the form of a square. At the 1st trial he had 130 sovereigns over; but when he enlarged the side of the square by 3 sovereigns, there only remained 31. How many sovereigns had he?

Ans. 355.

164. It is required to find a number, such, that if the 2 numbers a and b, be added to it, the difference of the squares of these sums may be $= d$. What number is it?

Ans. $\frac{d - a^2 + b^2}{2\,(a-b)}$.

Are the two preceding problems included in this one?

165. Determine the sizes of 3 wine-casks from the following conditions: When the 1st empty cask is filled from the 2nd full cask, there only remains in the 2nd $\frac{2}{9}$ of the wine; if the 2nd empty cask be filled from the 3rd full one, then there remains in the 3rd only $\frac{1}{4}$ of the wine; but if we wished to fill the 3rd empty cask from the 1st full one, there would not be enough by 50 quarts. How much does each of the casks contain?

Ans. The 1st 70, the 2nd 90, the 3rd 120 quarts.

166. A person has 4 wine-casks of different sizes. If, when he fills the 2nd empty cask from the 1st full one, there remains in the 1st only $\frac{4}{7}$ of the wine; if, when he fills the 3rd empty cask from the 2nd full one, then there is left in the 2nd only $\frac{1}{4}$ of the wine; but, if, when he

fills the 4th empty cask from the 3rd full one, then only $\frac{3}{16}$ of the 4th is filled ;and if he wished to fill the 3rd and 4th empty casks from the 1st full one, then these would not only be filled, but he would have 15 quarts over. How many quarts does each of the casks contain ?

Ans. The 1st 140, the 2nd 60, the 3rd 45, and the 4th 80 quarts.

XVI. PROBLEMS FOR SIMPLE EQUATIONS WITH MORE UNKNOWN QUANTITIES.

1. Required 2 numbers, whose sum is 70, and whose difference is 16 ? *Ans.* 43 and 27.

2. Required to find 2 numbers whose sum $= a$, and whose difference $= b$. How may these numbers be expressed ?

Ans. The 1st $= \frac{a + b}{2}$, the other $= \frac{a - b}{2}$.

3. Two purses together contain 300 sovereigns. If we take 30 sovereigns out of the 1st and put it into the 2nd then there is the same sum in each. How much does each contain ?

Ans. The 1st 180, the 2nd 120.

4. A says to B, give me 100*l.* and then I shall have as much as you. No, says B to A, give me rather 100*l.* and then I shall have twice as much as you. How much has each ? *Ans.* A 500*l.* and B 700*l.*

5. A person has two snuff-boxes. If he put 8*l.* into the 1st, then it is half as valuable as the other. But if he take these 8*l.* out of the 1st, and put them into the 2nd, then the latter is worth three times as much as the former. What is the value of each?

Ans. The 1st is worth 24*l.*, the 2nd 64*l.*

6. A and B possess together a fortune of 570*l.* If A's fortune were three times, and B's 5 times as great as each really is, then they would have together 2350*l.* How much has each? *Ans.* A 250*l.*, B 320*l.*

7. Find 2 numbers of the following properties. When the one is multiplied by 2, the other by 5, and both products added together, the sum is = 31; on the other hand, if the 1st be multiplied by 7, the 2nd by 4, and both products added together, we shall obtain 68.

Ans. The 1st is 8, the 2nd is 3.

8. If one of 2 numbers be multiplied by a, the other by b, the sum of the products $= k$; but if the 1st be multiplied by a', the 2nd by b', then the sum of the products $= k'$. How can these numbers be expressed?

Ans. By $\frac{b'k+bk'}{ab'-a'b}$, $\frac{ak'-a'k}{ab'-a'b}$.

9. Two numbers are given by the following data. If the 1st be increased by 4, it will be $3\frac{1}{4}$ times as great as the 2nd; but if the 2nd be increased by 8, then it will be half as great as the 1st. What are the numbers?

Ans. 48 and 16.

10. When the 1st of 2 numbers is increased by a, it becomes m times as great as the 2nd; but when the 2nd

is increased by b, then it is n times as great as the 1st. How are these numbers expressed?

Ans. The 1st $= \frac{a+mb}{mn-1}$; the 2nd $= \frac{b+na}{mn-1}$.

11. Said a man to his father, "how old are we?" "Six years ago," answered the latter, "I was $\frac{1}{3}$ more than 3 times as old as you; but 3 years after this I was so old, that I was obliged to multiply your age by $2\frac{1}{6}$, in order to obtain my own." What is the age of each?

Ans. The father 36, the son 15 years.

12. A and B jointly have a fortune of 9800*l.* A invests the 6th part of his property in business, and B the 5th part, and still each has the same sum remaining. How much has each? *Ans.* A has 4800*l.*, B 5000*l.*

13. A owes 1200*l.* B 2550*l.*, but neither has enough to pay his debts. Lend me, said A to B, the 8th part of your fortune, and I shall be enabled to pay my debts. B answered: I can discharge my debts if you will lend me the 6th part of yours. What was the fortune of each?

Ans. A's fortune is 900*l.*, and that of B 2400*l.*

14. A capitalist borrows 8000*l.* on favourable conditions, because he has an opportunity of laying 23000*l.* out to higher interest, and he has an overplus of 905*l.* of interest yearly. Under the same conditions he borrows, on the one hand, 9400*l.*, and on the other, lends 17500*l.*; this brings him in an overplus of 539$\frac{1}{2}$*l.* in interest yearly. At what rate of interest did he borrow and lend money?

Ans. At $4\frac{1}{2}$ and $5\frac{1}{2}$ per cent.

15. A person has two large pieces of ice, whose weight

is required. It is known that $\frac{2}{3}$ths of the 1st piece weighs 96lbs. less than $\frac{3}{4}$ths of the other piece, and that $\frac{5}{8}$ths of the other piece weighs exactly as much as $\frac{4}{9}$ths of the 1st. How much does each of the pieces weigh?

Ans. The 1st weighs 720, the 2nd 512lbs.

16. A cistern containing 210 buckets may be filled by 2 pipes. By an experiment in which the 1st was open 4, and the 2nd 5 hours, 90 buckets of water were obtained. By another experiment, when the 1st was open 7, and the other $3\frac{1}{2}$ hours, 126 buckets were obtained. How many buckets does each pipe discharge in an hour? And in what time will the cistern be full, when the water flows from both pipes at once?

Ans. The 1st pipe discharges 15, and the 2nd 6 buckets; it will require 10 hours to fill the cistern.

17. A person has 500 coins, consisting of sovereigns and shillings, their value amounts to 334*l.* 14*s.* How many has he of each coin?

Ans. 326 sovereigns, 174 shillings.

18. A person has two kinds of goods, 8lbs. of the 1st and 19lbs. of the 2nd cost together 18*l.* 4*s.* 2*d.*: farther, 20lbs. of the 1st and 16lbs. of the 2nd cost together 25*l.* 16*s.* 8*d.* How much does the lb. of each article cost?

Ans. 15*s.* 10*d.* and 12*s.* 6*d.*

19. 15 Silesian and 33 Leipzig ells together are equal to $39\frac{1}{2}$ Brabant ells; farther, 24 Silesian and 55 Leipzig ells are together equal to 65 Brabant ells. What proportion, according to these conditions, do the Silesian and

Leipzig ells bear to the Brabant? Farther, what the Silesian to the Leipzig ell? And by how many per cent do the two last differ from one another?

Ans. The Silesian ell is to the Brabant ell as 5 to 6, the Leipzig to the Brabant as 9 to 11, the Silesian to the Leipzig as 55 to 54; and the Silesian ell is $1\frac{23}{27}$ per cent longer than the Leipzig.

20. $17\frac{1}{2}$ Dantzic and 19 Berlin feet make together as much as $34\frac{3}{4}$ Rhenish feet; farther, 5 Dantzic and $9\frac{1}{2}$ Berlin feet are equal to $13\frac{53}{56}$ Rhenish. According to these statements, what proportion do the Dantzic and the Berlin feet bear to the Rhenish? What relation does the Dantzic foot bear to the Berlin? And by how many per cent do these two last differ?

Ans. The Dantzic foot is to the Rhenish as 32 to 35, the Berlin is to the Rhenish as 75 to 76, the Dantzic to the Berlin as 2432 to 2625, and the Berlin foot is $7\frac{569}{608}$, or about $7\frac{14}{15}$ per cent, longer than the Dantzic.

21. 40 French miles, when reduced to geographical or German miles, amount to $12\frac{1}{2}$ such miles more than 53 English. Ten French and $26\frac{1}{2}$ English miles are together equivalent to $11\frac{3}{4}$ German miles. At this rate, what proportion do the French and English bear to the German? And what relation have the French to the English?

Ans. The French mile is to the German as 3 to 5, the English is to the German as 23 to 106, and the French to the English as 318 to 115.

22. A person exchanged 250 Frederick d'ors for ducats, and received for them 439 ducats and 14 gros. He exchanged 160 Frederick d'ors more at the same rate, and received for them 281 ducats and 6 gros. What was each of these gold coins calculated at?

Ans. The Frederick d'or at 5 dollars 10 gros., and the ducat at 3 dollars 2 gros.

23. A traveller said: "I have travelled through Germany, France, and England, and have expended in these three countries the sum of 8325 dollars; viz. in Germany 1520 dollars, in France 7540 francs, and in England 820 pounds sterling." As he was asked to fix by this calculation the value of the different kinds of money, he gave for answer, that 5 pounds sterling amounted to 3 dollars more than 108 francs. What was each coin computed at?

Ans. The pound sterling at 6 dollars, and the franc at 6 gros.

24. A person has two horses, and also two saddles, one of which costs 50*l.*, the other 2*l.* If he place the best upon the 1st horse, and the worst upon the 2nd, then the latter is worth 8*l.* less than the other. But if he put the worst saddle upon the 1st, and the best upon the 2nd horse, then the latter is worth $3\frac{3}{4}$ times as much as the former. What is the value of each horse?

Ans. The 1st 30*l.*, the 2nd 70*l.*

25. There is a fraction such, that if 1 be added to the numerator, its value $= \frac{1}{3}$, and if 1 be added to the denominator, its value $= \frac{1}{4}$. What fraction is it?

Ans. $\frac{4}{15}$.

26. It is required to find a fraction such, that if 3 be subdivided from the numerator and denominator, it is changed to $\frac{1}{4}$, and if 5 be added to the numerator and denominator, it becomes $\frac{1}{2}$. What fraction is it?

Ans. $\frac{7}{19}$.

27. B has lent out at interest 12600*l.* more than A, and obtains 1 per cent more for his money, on which account his interest amounts yearly to 750*l.* more than A's. C has lent 3000*l.* more than A, and also at 2 per cent higher interest, he obtains for this sum 360*l.* interest more than A. How much has each lent? and at what interest?

Ans. A has lent 10000*l.*, B 22600*l.*, C 13000*l.*; A at 4 per cent, B at 5, and C at 6.

28. A company expended a certain sum in an inn, and each the same (to be equally divided amongst them). Had there been 5 persons more, and had each expended 2*s.* 6*d.* more, the reckoning would have come to 6*l.* 18*s.* 10*d.* more; but had there been 3 persons less, and each expended 1*s.* 8*d.* less, then it would have amounted to 3*l.* 8*s.* 4*d.* less. How many were there in the company, and what did each spend?

Ans. The company consisted of 14 persons, and each spent 16*s.* 8*d.*

29. A work is so printed, that each page contains a certain number of lines, and each line a certain number of letters. If we wished each page to contain 3 lines more, and each line 4 letters more, then there would be 224 letters more than before; but if we wished to have 2 lines less in each page, and 3 letters less in each line, then the

page would contain 145 letters less. How many lines are there in each page, and how many letters in each line?

Ans. 29 lines, and 32 letters.

30. Required to find 2 numbers such, that if one be increased by a, the other by b, the product of these 2 sums exceeds the product of the 2 numbers themselves by c; if, on the other hand, the one be increased by a', the other by b', the product of these sums exceeds the product of the numbers themselves by c'. How are these numbers expressed?

Ans. They are $\frac{a'c-ac'+aa'(b'-b)}{a'b-ab'}$ and

$$\frac{bc'-b'c+bb'(a-a')}{a'b-ab'}.$$

Are the 3 preceding problems included in this one? And what values must be assigned to the letters a, b, c, a', b', c', so that the former problems may be solved?

31. Not long ago, says a person, the quarter of wheat was 1*l.*, and the quarter of rye 17*s.* 6*d.* cheaper than they are now; then the price of the wheat was to the price of the rye as 10 to 7; their present prices are as 4 to 3. What is the price of the quarter of each grain?

Ans. The quarter of wheat costs 3*l.* 10*s.*: the quarter of rye 2*l.* 12*s.* 6*d.*

32. A person has 2 casks, and a certain quantity of wine in each. In order to have an equal quantity in each, he pours as much out of the 1st cask into the 2nd as it already contains, then again he pours as much out of the 2nd into the 1st as it now contains, and lastly, again, as much from the 1st into the 2nd as there is still re-

remaining in it. At last he has 16 quarts of wine in each cask. How many quarts did they contain originally?

Ans. In the 1st 22 quarts, in the 2nd 10 quarts.

33. When, in the preceding problem, there remain at last a quarts in each cask: how many quarts must they have contained originally?

Ans. The 1st $\frac{11}{8}a$, in the 2nd $\frac{5}{8}a$ quarts.

34. A wine-merchant has 2 kinds of wine. If he mix 3 quarts of the best with 5 quarts of the worst, he can sell the mixture for 17*s.* 1*d.* a quart. But if he mix $3\frac{3}{4}$ quarts of the best with $7\frac{1}{2}$ quarts of the worst, then he can sell the quart for 16*s.* 8*d.* exactly. What does each wine cost a quart?

Ans. The best wine 1*l.* 3*s.* 4*d.* a quart, the worst 13*s.* 4*d.* a quart.

35. In general, let a quarts of the 1st wine mixed with b quarts of the 2nd, be worth, on an average, c shillings: farther, let f quarts of the 1st mixed with g quarts of the 2nd, average worth h shillings. What does each wine cost a quart?

Ans. The price of the 1st is $\frac{(a+b)cg-(f+g)bh}{ag-bf}$ shillings, the price of the 2nd is $\frac{(a+b)cf-(f+g)ah}{bf-ag}$ shillings.

36. 37 lbs. of tin lose 5 lbs. in water, and 23 lbs. of lead lose 2 lbs. in water; a composition of tin and lead weighing 120 lbs. loses 14 lbs. in water. How much does this composition contain of each metal.

Ans. 74 lbs. of tin, and 46 lbs of lead.

37. 21 lbs. of silver lose 2 lbs. in water, and 9 lbs. of copper lose 1 lb. in water. If a composition of silver and copper weighing 148 lbs. loses $14\frac{2}{3}$ lbs. in water, how much does it contain of each?

Ans. 112 lbs. of silver, and 36 lbs. of copper.

38. A given piece of metal, which weighs p lbs., loses a lbs. in water. This piece, however, is composed of 2 other metals, which call A and B; of these we know, that p lbs. of A lose b lbs. in water, and p lbs. of B lose c lbs. How much does this piece contain of each metal?

Ans. $\frac{(c-a)p}{c-b}$ lbs. of A, and $\frac{(a-b)p}{c-b}$ lbs. of B.

39. According to Vitruvius, Hiero, king of Syracuse's crown weighed 20 lbs., and lost $1\frac{1}{4}$ lbs. nearly in water. Let it be assumed, that it consisted of gold and silver only, and that 19·64 lbs. of gold lose 1 lb. in water, and 10·5 lbs. of silver, in like manner, lose 1 lb.: how much gold and how much silver did this crown contain?

Ans. 14·77... lbs. of gold, and 5·22... lbs. of silver.

Is this problem contained in the preceding one? And what must be here assumed for p, a, b, c?

40. Lead is 11·324 heavier than water; cork only weighs 0·24 times as much as water; again, fir weighs 0·45 times as much as water. A person wishes to combine a piece of lead with a piece of cork, so that he may obtain a body of 80 lbs. weight, exactly of the same weight as a piece of fir of the same size (which, consequently, will float). How much lead and cork must be combined together?

Ans. 38·14... lbs. of lead with 41·85... lbs. of cork.

41. Two different kinds of matter, one of which is p, and the other p' times as heavy as water, are to be so united, that the bodies which arise from their union may on an average be p'' times as heavy as water, and weigh q lbs. In order to effect this, how many lbs. must be taken of each substance?

Ans. $\frac{qp(p'-p'')}{p''(p'-p)}$ lbs. of the 1st, and $\frac{qp'(p''-p)}{p''(p'-p)}$ lbs. of the 2nd substance.

Between what limits $\therefore$ must p'' lie, that the problem, as it is here given, may be possible?

42. It is required to find 2 numbers, whose difference, sum, and product, are to one another as the numbers 2, 3, 5, consequently, whose difference is to their sum as 2 to 3, and whose sum is to their product as 3 to 5. What numbers are they? *Ans.* 2 and 10.

43. Find 3 numbers, whose sum is m, and whose product is n times as great as their difference. What numbers are they?

Ans. $\frac{2n}{m-1}$ and $\frac{2n}{m+1}$.

44. Let the sum of 2 numbers be 13, and the difference of their squares 39. What numbers are they?

Ans. 5 and 8.

45. The sum of 2 numbers is $= a$, the difference of their squares $= b$. What are the numbers?

Ans. $\frac{a^2+b}{2a}$ and $\frac{a^2-b}{2a}$.

46. The sum of 2 numbers $= a$, the quotient arising from the divisor of the one by the other, is $= b$. Find these numbers?

Ans. $\frac{a}{b+1}$ and $\frac{ab}{b+1}$.

47. A person was asked his own, his father's, and grand-father's ages. He answered, "My age and my father's amount to 56 years, my father's age and my grand-father's to 100 years, mine and my grand-father's to 80 years." What was the age of each?

Ans. He is himself 18 years, his father 38, and his grand-father 62.

48. The sums of 3 numbers, taken two and two, are a, b, c. What are these numbers?

Ans. $\frac{a+b-c}{2}, \frac{a+c-b}{2}, \frac{b+c-a}{2}$.

49. A, B, C, owe 2190*l.* amongst them, and no one of them can pay this sum alone. But when they unite, it can be done in the following way: by B's putting $\frac{5}{7}$ of his property to all A's; or, by C's putting $\frac{5}{9}$ of his property to that of B; or, by A's adding $\frac{2}{3}$ of his property to that of C. How much did each possess?

Ans. A 1530*l.*, B 1540*l.*, C 1170*l.*

50. A and B possess together only $\frac{2}{3}$ of the property of a third C; B and C have together 6 times as much as A; were B 680*l.* richer than he actually is, then he would have as much as A and C together. How much has each? *Ans.* A has 200*l.*, B 360*l*, and C 840*l.*

51. I have 3 purses lying before me, each of which contains a certain sum of money. If I take 20*l.* out of the 1st and put them into the 2nd, then the latter contains 4 times as much as the former. If, on the other hand, I take 60*l.* out of the 2nd and put them into the 3rd, then this contains $1\frac{3}{4}$ times as much as there is in

the 2nd. Again, if I take 40*l.* out of the 3rd and put them into the 1st, then the 3rd contains $2\frac{7}{8}$ times as much as the 1st. What were the contents of each purse?

Ans. In the 1st 120*l.*, in the 2nd 380*l.*, and in the 3rd 500*l.*

52. A, B, C compare their fortunes. A says to B, give me 700*l.* of your money, and I shall have twice as much as you retain; B says to C, give me 1400*l.* and I shall have thrice as much as you have remaining; C says to A, give me 420*l.* and then I shall have 5 times as much as you retain. How much has each?

Ans. A 980*l.*, B 1540*l.*, C 2380*l.*

53. Find 3 numbers of the following properties. If we subtract 4 from the 1st and add as many to the 2nd, then the remainder is to the sum as 1 to 2. If we subtract 10 from the 2nd and add the same number to the 3rd, then the remainder is to the sum as 3 to 10. But if we subtract 5 from the 1st, and add this number to the 3rd, then the remainder is to the sum as 3 to 11. What numbers are they? *Ans.* 20, 28, and 50.

54. A, B, C together possess 1820*l.* If B give A 200*l.* of his money, then A will have 160*l.* more than B; but if B receive 70*l.* from C, then both will have the same sum. How much has each?

Ans. 400*l.*, 640*l.*, and 780*l.*

55. Three persons jointly spent a certain sum; but neither of the 3 is able to pay it of himself. A, therefore, says to B, give me the 4th part of your money, and then I can pay it alone. B says to C, give me the 8th part of your money and then I can also pay it; then says C

to A, I shall also be able to pay it if I receive from you the half of your money, although I, at present, only possess 4*l.* How much did they spend? And how much have A and B?

Ans. They spent 6*l.* 10*s.*; A has 5*l.* and B 6*l.*

56. A person has 3 pieces of silver of different alloy, viz. 15, 10, and 9 parts, out of 16 pure. If he melt the 15 with the 10, then there will arise a composition of $11\frac{2}{3}$ parts pure. The silver arising from the mixing the 15 and 10 together will be of the same purity. All three pieces together weigh 34 lbs. How much does each piece weigh by itself?

Ans. The 15 weighs eight lbs. the 10 sixteen, and the 9 ten lbs.

57. A person 3 warehouses, each of which contains 3 kinds of grain, viz. wheat, rye, and barley. The 1st warehouse contains 24 quarters of wheat, 9 quarters of rye, and 15 quarters of barley; the 2nd 9 quarters of wheat, 30 quarters of rye, and 21 quarters of barley; the 3rd 18 quarters of wheat, 27 quarters of rye, and 39 quarters of barley. The value of the 1st warehouse is 734*l.*; the value of the 2nd 812*l.*; and the value of the 3rd 1130*l.* What was the value of a quarter of each grain?

Ans. The quarter of wheat is worth 56*l.*; the quarter of rye 42*l.*; and the quarter of barley 32*l.*

58. A, B, C purchase coffee, sugar, and tea at the same prices. A pays 11*l.* 12*s.* 6*d.* for $7\frac{1}{2}$ lbs. of coffee, 3 lbs. of sugar, and $2\frac{1}{4}$ lbs. of tea; B pays 16*l.* 5*s.* for 9 lbs. of coffee, 7 lbs. of sugar, and 3 lbs. of tea; C pays

12*l.* 5*s.* for 2 lbs. of coffee, 5½ lbs. of sugar, and 4 lbs. of tea. What does each cost a lb.?

Ans. The coffee 15*s.*; the sugar 10*s.*; and the tea 2*l.*

59. Three masons, A, B, C, are to build a wall. A and B jointly could build this wall in 12 days; B and C could accomplish it in 20 days; but A and C would do it in 15 days. What time would each take to do it in? And in what time will they finish it if they all three work together?

Ans. A requires 20 days, B 30, and C 60; all 3 together require 10 days.

60. Three labourers are employed in a certain work. A and B would, together, complete this work in a days; A and C require b days, but B and C, c days. What time would each require singly to accomplish it in, supposing, under all circumstances, that each does the same quantity of work? And in what time would they finish it if they all three worked together?

Ans. A requires $\frac{2abc}{cb+ac-ab}$ days, B $\frac{2abc}{bc+ab-bc}$ days, and C $\frac{2abc}{ab+ac-bc}$ days. Jointly they require $\frac{2abc}{ab+ac+bc}$ days.

61. A cistern may be filled by 3 pipes, A, B, C. By the pipes A and B it could be filled in 70 minutes, by the pipes A and C in 84 minutes, and by the pipes B and C in 140 minutes. What time will each pipe take

to do it in? And in what time will the reservoir be filled, if the water flow into it from all the 3 pipes at once?

Ans. A requires 105 minutes, B 210, and C 420; all 3 pipes together require an hour.

Is this problem also included (under) in the 60th? And, if so, how must the conditions be changed in the last, in order that it may be quite similar?

62. A person has 3 pieces of metal, each of which consists of gold, silver, and copper. The 1st contains 5 oz. of gold, 15 oz. of silver, and 30 oz. of copper; the 2nd contains 20 oz. of gold, 28 oz. of silver, and 48 oz. of copper; the 3rd contains 12 oz. of gold, 39 oz. of silver, and 24 oz. of copper. Now, he wishes to take some from each, and to melt all into a mass, in order to obtain a composition consisting of 10 oz. of gold, 23 oz. of silver, and 26 oz. of copper. How much then must he take of each?

Ans. Of the 1st 10 oz., of the 2nd 24 oz, and of the 3rd 25 oz.

63. Three soldiers in a battle got 96*l.* booty, which they wish to share equally. In order to do this, A, who had obtained most, gives A and B as much as they already had; in the same manner B then divided with A and C, and after this C with A and B. If, then, by these means, the intended division be effected: how much booty did each soldier get?

Ans. A 52*l.*, B 28*l.*, and C 16*l.*

64. In the 3 drawers of my cup-board there are, altogether, a sum of 162*l.* In order that there may be

the same sum in all the drawers, I take out of the 1st as much as is necessary, and put into each of the other 2 the half of what they already contained. I then take out of the 2nd, and afterwards out of the 3rd drawer, and put each time into the 2 other drawers the half of what they already contained. If, by these means, I have actually attained my object, how much did each drawer contain at first?

Ans. In the 1st 70*l.*, in the 2nd 52*l.* and in the 3rd 40*l.*

65. A, B, C play Faro. In the 1st game A has the bank, B and C stake the 3rd part of their money and win. In the 2nd game B has the bank; A and C stake the 3rd part of their money and also win. Then C takes the bank; A and B stake the 3rd part of their money, and this time also the banker loses. After the 3rd game they count their money, and find that they all have the same sum, viz. 64*l.* each. How much had they before they began to play?

Ans. A 75*l.*, B 63*l.*, C 54*l.*

66. A, B, C, D, E play together on the conditions, that he who loses shall give to all the rest as much as they have already. First A loses, then B, then C, then D, and at last also E. All lose in turn, and yet at the end of the 5th game they all have the same sum, viz. each 32*l.* How much had each before he began to play?

Ans. A 81*l.*, B 41*l.*, C 21*l.*, D 11*l.*, and E 6*l.*

67. 2652*l.* are divided amongst 3 regiments, in such a way, that each man of that regiment which contains most, receives 1*l.*, and the remainder is divided equally

amongst the men of the other 2 regiments. Were the pound adjudged to the 1st regiment, then each man of the 2 remaining regiments would receive 10*s.*; if we give the pound to the 2nd regiment, then each man of the 2 remaining regiments would receive 6*s.* 8*d.*; lastly, if it were given to the 3rd regiment, then each man of the remaining regiments would only receive 5*s.* How strong was each of the 3 regiments?

Ans. The 1st 780 men, the 2nd 1716, the 3rd 2028.

68. It is required to determine 3 numbers from the following data. If the 1st be added to m times the others, then the sum $= a$; if the 2nd be added to the m' times the others, then the sum $= a'$; but if the 3rd be added to m'' times the others, then the sum $= a''$. What numbers are they?

Ans. $\frac{m}{m-1}+\frac{m'}{m'-1}+\frac{m''}{m''-1}=\mathrm{A}, \frac{a}{m-1}+\frac{a'}{m'-1}+\frac{a''}{m''-1}=\mathrm{B}$; the 3 numbers are $\frac{1}{m-1}\left(\frac{mB}{A-1}-a\right)$, $\frac{1}{m'-1}\left(\frac{m'B}{A-1}-a\right)$, $\frac{1}{m''-1}\left(\frac{m''B}{A-1}-a''\right)$, and their sum $=\frac{B}{A-1}$.

69. There are 3 numbers whose sum is if 7 be subtracted from the 1st and 2nd, the remainders are to one another as 5 to 3; on the other hand, if 3 be subtracted from the 2nd and 3rd, then the remainders are to one another as 11 to 9. What numbers are they?

Ans. 37, 25, 21.

70. Required to find 3 numbers, which possess the following properties: if 6 be added to the 1st and 2nd, the sums are to one another as 2 to 3; if 5 be added to the 1st and 3rd, then the sums are to one another as 7 to 11; but if 36 be subtracted from the 2nd and 3rd, the remainders are to one another as 6 to 7. What numbers are they? *Ans.* 30, 48, 50.

71. A certain number consists of 3 digits, which are in arithmetical progression. If this number be divided by the sum of its digits (consequently, without regard to the values which they have by reason of their positions), the quotient is 48; but if we subtract 198 from this number, we obtain a number which has the same digits as the one sought, but in an inverted order. What number is this? *Ans.* 432.

XVII. PROBLEMS FOR QUADRATIC EQUATIONS WITH ONE AND MORE UNKNOWN QUANTITIES.

1. What number is it, whose half multiplied by its 3rd part, gives 864? *Ans.* 72.

2. What number is it, whose 7th and 8th parts multiplied together, and the product divided by 3, gives the quotient $298\frac{2}{3}$? *Ans.* 224.

3. It is required to find a number, such, that if we 1st add it to 94, then subtract it from 94, and afterwards

multiply this remainder by the former sum, the product may be 8512. What number is it? *Ans.* 18.

4. What numbers are they, whose product is 750, and quotient $3\frac{1}{3}$? *Ans.* 50 and 15.

5. The product of 2 numbers $= a$, their quotient $= b$. By what formulæ can these be expressed?

Ans. $\sqrt{ab}$ and $\sqrt{\frac{a}{b}}$.

6. Find 2 numbers, the sum of whose squares $=$ 13001, and the difference of their squares $=$ 1449.

Ans. 85 and 76.

7. The sum of the squares of 2 numbers $= a$, the difference of their squares $= b$. What formulæ will express them?

Ans. $\sqrt{\frac{a+b}{2}}$ and $\sqrt{\frac{a-b}{2}}$.

8. What numbers are to one another as 3 to 4, and the sum of whose squares $=$ 324900?

Ans. 342 and 456.

9. What numbers are as m to n, and the sum of whose squares $= b$?

Ans. $\frac{m\sqrt{b}}{\sqrt{(m^2+n^2)}}, \frac{n\sqrt{b}}{\sqrt{(m^2+n^2)}}$.

10. What numbers are to one another as m to n, and the difference of whose squares $= b$?

Ans. $\frac{m\sqrt{b}}{\sqrt{(m^2-n^2)}}, \frac{n\sqrt{b}}{\sqrt{(m^2-n^2)}}$.

11. A certain capital bears 4 per cent; if we multiply the number of pounds in the capital by the number of

pounds in the interest for 5 months, we obtain 117041⅔. What is the capital? *Ans.* 2650.

12. A person has 3 kinds of goods, which together cost 230*l.* 4*s.* 2*d.* The lb. of each article costs as many ten-pences as he has lbs. of that article; but he has ⅓ more of the 2nd kind than he has of the 1st, and 3½ times as much of the 3rd as he has of the 2nd. How many lbs. has he of each article?

Ans. Of the 1st 15 lbs., of the 2nd 20 lbs., and of the 3rd 70 lbs.

13. A person has on hand a small quantity of a certain article. On my asking him how many lbs. there were of it, he gave me for answer, "If I sold the lb. for 2⅗ times as many ten-pences as there are lbs., I should lose by that means exactly as much above 6*l.* 5*s.* 10*d.* as I should lose under this sum, were I to sell the lb. for half as many ten-pences as there are lbs." How many lbs. had he? *Ans.* 10.

14. I have a certain number in my thoughts; this I multiply by 2⅓, add 7 to the product, multiply this sum by 8 times the number; I then divide by 14, and subtract from the quotient 4 times the number; I then obtain 2352. What number is it? *Ans.* 42.

15. Required to find 3 numbers, such, that the product of the 1st and 2nd $= a$, the product of the 1st and 3rd $= b$, and the sum of the squares of the 2nd and 3rd $= c$?

Ans. $\sqrt{\frac{a^2+b^2}{c}}$, $a\sqrt{\frac{c}{a^2+b^2}}$, $b\sqrt{\frac{c}{a^2+b^2}}$.

16. What 3 numbers are they, which multiplied two and two, and each product divided by the 3rd, the quotients are a, b, c? *Ans.* $\sqrt{ab}$, $\sqrt{ac}$, $\sqrt{bc}$.

17. What 3 numbers possess these properties, that the product of the 1st and 2nd, the product of the 2nd and 3rd, and the product of the 3rd and 1st, give respectively the numbers a, b, c? *Ans.* $\sqrt{\frac{ac}{b}}$, $\sqrt{\frac{ab}{c}}$, $\sqrt{\frac{bc}{a}}$.

18. What 5 numbers possess these properties, that if each, beginning with the 1st, be multiplied by the one which succeeds (follows) it, but the last again by the 1st, the products a, b, c, d, e, are obtained? *Ans.* $\sqrt{\frac{ace}{bd}}$, $\sqrt{\frac{abd}{ce}}$, $\sqrt{\frac{bce}{ad}}$, $\sqrt{\frac{acd}{be}}$, and $\sqrt{\frac{bde}{ac}}$.

19. But if, instead of 5, seven numbers be required, and the products be a, b, c, d, e, f, g; what numbers are they then? *Ans.* $\sqrt{\frac{aceg}{bdf}}$, $\sqrt{\frac{abdf}{ceg}}$, &c. &c.

Similar expressions may be found for every odd number of the numbers sought, but only under certain conditions for an even number. Why so?—And what condition must the numbers a, b, c, d, &c. be subject to, in order that it may be possible?

20. There are two numbers, one of which is greater than the other by 8, and whose product is 240. What numbers are they? *Ans.* 12 and 20.

21. The sum of 2 numbers $= a$, their product $= b$. What numbers are they?

Ans. $\frac{a+\sqrt{(a^2-4b)}}{2}$ and $\frac{a-\sqrt{(a^2-4b)}}{2}$.

22. It is required to find a number, whose square exceeds its simple power by 306? *Ans.* 18.

23. It is required to find a number, such, that if we multiply its 3rd part by its 4th, and to the product add 5 times the number required, this sum exceeds the number 200 by as many as the number sought is less than 280?

Ans. 48.

24. A person who was asked his age, answered, "My mother was 20 when I was born; her age multiplied by mine, exceeds our united ages by 2500." What was his age? *Ans.* 42.

25. A merchant has 2 kinds of tea of different weight and price. The weight of the 1st kind is to the weight of the 2nd, as 4 to 3. The lb. of the 1st costs half as many ten-pences, as it weighs in lbs.; the 2nd costs 5*s.* a lb. less than the 1st. The value of the tea is 2*l.* 6*s.* 8*d.* How much does each kind weigh?

Ans. The 1st 80lbs., the 2nd 60lbs.

26. Determine the fortunes of 3 persons, A, B, C, from the following data:—For every 5 pounds which A possesses, B has 9, and C 10. Farther, if we multiply A's money (expressed in pounds, and considered merely as a number) by B's, and B's money by C's, and add both products to the united fortunes of all 3, we shall get 8832. How much had each? *Ans.* A 40, B 72, C 80.

27. A person buys some pieces of cloth at equal prices for 60*l.* Had he got 3 more pieces for the same sum, each piece would have cost him 1*l.* less. How many pieces did he buy? *Ans.* 12.

28. A charitable person divides a sum of 36*l.* in equal shares amongst the poor of a small town. But as 6 of those whom he thought of relieving stood no longer in need of assistance, each of the remaining paupers had for his share 10*s.* 8*d.* more than he otherwise would have had. How many paupers were there at first? *Ans.* 54.

29. A person dies, leaving children and a fortune of 46800*l.*, which, by the will, is to be divided equally amongst them. It happens, however, that immediately after the death of the father, two of his children also die. If, consequently, each child receives 1950*l.* more than he or she was entitled to by the will, how many children were there? *Ans.* 8 children.

30. Required to find a number, such, that if a given number c be divided by it, and also by a number greater than it by a, the difference of the two quotients $= d$. What number is it? *Ans.* $-\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + \frac{ac}{d}\right)}$.

Are the 3 preceding problems included in this last?

31. Twenty persons, men and women, together spend 48*l.* at a tavern; viz. the men 24*l.*, and the women the same sum. Now, on inspecting the bill it is found, that the men have to pay 1*l.* each more than the women. How many men, therefore, were there in company? *Ans.* 8.

32. A person buys a horse, and pays a certain sum for it, he afterwards sells it again for 144*l.*, and gains exactly as much per cent as the horse had cost him. How much did the horse cost? *Ans.* 80*l.*

33. A merchant sends for a piece of stuff, and pays a certain sum for it, besides 4 per cent for carriage. He sells it again for 390*l.*, and gains by the bargain as much per cent as the 12th part of the purchase-money amounts to. What did he buy it for? *Ans.* 300*l.*

34. Two drapers cut, each of them a certain number of ells, from a piece of cloth; one, however, 3 ells less than the other, and jointly receive for them 35*l.* "I should have received," said the first to the other, "at my price, 24*l.* for your cloth." "I must admit," the other answered him, "that at my low price I should have got no more than 12*l.* 10*s.* for your cloth." How many ells had each?

Ans. The one 15, the other 18; or the one 5, the other 8.

35. Two travellers, A and B, set out at the same time from two different places, C and D, A from C to D, and B from D to C. It appears that A has already gone 30 miles more than B, and according to the rate at which they travel, A calculates that he can reach the place D in 4 days, and that B can arrive at the place C in 9 days. What is the distance between C and D?

Ans. 150 miles.

36. In the preceding problem, let d be the distance which A had travelled more than B; a the time which A requires to finish the remainder of his journey, and b the time which B requires, in order to finish his. What expression will give the distance between C and D?

Ans. $\frac{d(\sqrt{b}+\sqrt{a})}{\sqrt{b}-\sqrt{a}}$.

37. Two retailers jointly invested 500*l.* in business, to which each contributed a certain sum; the one let his money remain 5 months, the other only 2, and each received 450*l.* capital and profit. How much did each advance?

Ans. One 200*l.*, the other 300*l.*

38. Two persons jointly invested 2000*l.* in business. One let his money remain 17 months, and received back in capital and profit 1710*l.*; the other allowed his money to remain 12 months, and received in capital and interest 1040*l.* How much did each advance?

Ans. One 1200*l.*, the other 800*l.*

39. What numbers are they whose sum is 41, and the sum of whose squares is 901? *Ans.* 26 and 15.

40. The sum of 2 numbers $= a$, the sum of their squares $= b$. What numbers are they?

Ans. $\frac{a+\sqrt{(2b-a^2)}}{2}$, and $\frac{a-\sqrt{(2b-a^2)}}{2}$.

When are these numbers imaginary?

41. What numbers are they whose difference is 8, and the sum of whose squares is 544?

Ans. 12 and 20.

42. What two numbers are they whose product is 255, and the sum of whose squares is 514?

Ans. 15 and 17.

43. Divide the number 16 into two parts, so that the product of the two parts, added to their squares, may be $= 208$. *Ans.* Into 4 and 12.

44. Into what two parts must 39 be divided, so that the sum of their cubes may be = 17199?

Ans. The two parts are 15 and 24.

45. A person was asked respecting his yearly income, he answered as follows: "My income is such, that if I add 1578*l.* to it, and also subtract 142*l.* from it, and extract the cube roots of the numbers thus obtained, the difference between the roots is 10." *Ans.* 150*l.*

46. What number is it, which added to its square root, = 1332? *Ans.* 1296.

47. What number is it which exceeds its square root by $48\frac{3}{4}$? *Ans.* $56\frac{1}{4}$.

48. There are two numbers, a and b, given; it is required to divide each of them into two such parts, that the one part of a is to one part of b as m to n, and that the product of the other two parts $= p$. How must they be divided?

Ans. Let $\frac{na+mb\sqrt{[(na-mb)^2+4mnp]}}{2mn} = A$, then

1 part of $a = mA$, and one part of $b = nA$.

49. Again. Let it be required, as in the preceding problem, to divide the two numbers a, b, so that the first parts may be to one another as m to n, but the sum of the squares of the two others $= s$. How then must they be divided?

Ans. Let $\frac{am+bn\pm\sqrt{\{(m^2+n^2)s-(an-bm)^2\}}}{m^2+n^2} = A$, then the 1st part of $a = mA$, and the 1st part of $b = nA$.

50. Find two numbers, whose difference added to the difference of their squares is 150; and whose sum added to the sum of their squares is 330. What numbers are they? *Ans.* 9 and 15.

51. What numbers are they, whose sum, product, and difference, of their squares are equal?

Ans. $\frac{3 \pm \sqrt{5}}{2}$, $\frac{1 \pm \sqrt{5}}{2}$.

52. There are three numbers in continual proportion; if we add them, their sum is 126; but if we multiply them together, the product is 13824. What numbers are they?

Ans. 6, 24, and 96.

53. It is required to find a number, consisting of 3 digits, such, that the sum of the squares of the digits, without considering their position, may be = 104; but the square of the middle digit exceeds twice the product of the other 2 by 4; farther, that if 594 be subtracted from the number sought, the 3 digits become inverted?

Ans. 862.

We must not always immediately infer that the quantities required to be found, are the unknown quotients in the calculation; we should otherwise not unfrequently hit upon higher equations than are necessary for the solution of the problem, and this we must try to avoid as much as possible. It is often better to seek first any combination of the quotients, as, for instance, the sum, the difference, the product, the sum of the squares, the difference of the squares, and so on, and hence to determine the quotients themselves. As this is a very important point in Algebra,

and one which cannot be too well observed, I shall now give a tolerable number of such problems; more of this kind will occur farther on.

54. Find two numbers, whose difference, multiplied by the difference of their squares, = 160; and whose sum, multiplied by the sum of their squares, gives the number 580.

Ans. The sum of the two numbers is 10, and their product 21; therefore the numbers themselves are 3 and 7.

55. A person wishes to find two numbers, such, that the sum and product of the numbers together amount to 34, and the sum of their squares exceeds the sum of the numbers themselves by 42. What numbers are they?

Ans. The sum of both numbers is 10, their product 24, and the numbers themselves are consequently 4 and 6.

56. If, in order to make the foregoing problem more general, a be put instead of 34, and b instead of 42: in this case, by what form will the numbers sought be expressed?

Ans. Let $-1 \pm \sqrt{(4b+8a+1)} = 2A$, $2A+1 \mp \sqrt{(4b+8a+1)} = 2B$, then the two numbers sought are $\frac{A+\sqrt{(A^2-4B)}}{2}$, $\frac{A-\sqrt{(A^2-4B)}}{2}$.

57. What two numbers are they, whose sum $= a$, and the sum of whose fourth powers $= b$?

Ans. Call the difference of the two numbers sought

d, then $d = \sqrt{\{-3a^2 \pm \sqrt{(8a^4+8b)}\}}$; the numbers themselves are consequently $\frac{a+d}{2}$, $\frac{a-d}{2}$.

58. The sum of two numbers is $=a$, the sum of their 5th powers $= b$. What numbers are they?

Ans. The product p of both numbers is $= \frac{1}{2}\left\{a^2 \pm \frac{\sqrt{a^5+4b}}{5a}\right\}$; therefore the numbers themselves are $\frac{1}{2}\left\{a^2 \pm \frac{\sqrt{a^5+4b}}{5a}\right\}$, $\frac{1}{2}\{a + \sqrt{(a^2-4p)}\}$, $\frac{1}{2}\{a - \sqrt{(a^2-4p)}\}$.

59. The sum of two numbers is $=a$, their products multiplied by the sum of their squares is $= b$. What numbers are they?

Ans. Let the products of the two numbers $=p$, if $p = \frac{1}{4}[a^2 \pm a^4 - 8b]$, then the numbers themselves are $\frac{1}{2}\{a + \sqrt{(a^2-4p)}\}$, $\frac{1}{2}\{a - \sqrt{(a^2-4p)}\}$.

60. The sum of two numbers added to the sum of their squares $= a$, m times the sum of their squares added to n times the product of the numbers $= b$. What numbers are they?

Ans. The sum s and the product p of both numbers are expressed by the equations $ns^2 + (n-2m)s = 2b + (n-2m)a$, $2p = s^2 + s - a$. Having determined from these s and p, then both the numbers themselves may be found by solving the equation $x^2 - sx + p = 0$. Each number, therefore, has four values.

61. In a geometrical proportion the sum of the means $= a$, the sum of the 2 extremes $= b$, and the sum of the squares of all 4 terms $= c$. What is the proportion?

Ans. The product of the 2 means, consequently of the 2 extremes, $= \frac{a^2+b^2-c}{4}$, therefore the required proportion is

$$\tfrac{1}{2}[b-\sqrt{(c-a^2)}] : \tfrac{1}{2}[a-\sqrt{(c-b^2)}]$$
$$= \tfrac{1}{2}[a+\sqrt{(c-a^2)}] : \tfrac{1}{2}[b+\sqrt{(b-a^2)}]$$

62. The difference between the means of a geometrical proportion $= a$, the difference of the two extremes $= b$, and the sum of the squares of all the four terms $= c$. What is the proportion?

Ans. The product of the two extremes, or of the two means, $= \frac{c-a^2-b^2}{4}$; $\therefore$ the proportion itself is

$$: \tfrac{1}{2}[-b+\sqrt{(c-a^2)}] : \tfrac{1}{2}[-a+\sqrt{(c-b^2)}] =$$
$$\tfrac{1}{2}[+a+(c-b^2)] : \tfrac{1}{2}[b+\sqrt{(c-a^2)}]$$

63. In a geometrical proportion the product of the two extremes, or means, $= a$, the sum of all four terms $= b$, and the sum of their squares $= c$. Find the proportion

Ans. For shortness-sake, let $\pm\sqrt{(8a+2c-b^2)} = A$; then $\frac{b-A}{2}$ is the sum of the two means, and $\frac{b+A}{2}$ the sum of the two extremes; consequently the proportion sought is $\tfrac{1}{4}[b+A-\sqrt{(2c-8a+2bA)}]$ $: \tfrac{1}{4}[b-A-\sqrt{(2c-8a-2bA)}] = \tfrac{1}{4}[(b-A+\sqrt{(2c-8a+2bA)}] : \tfrac{1}{4}[b+A+\sqrt{(2c-8a+2bA)}]$.

64. The product of the two extremes, or means, of a geometrical proportion $= a$, the difference between the

sum of the extremes and the sum of the means $= b$, and the sum of the squares of all four terms $= c$. What is the proportion?

Ans. Again; let $\pm\sqrt{(8a+2c-b^2)} = A$; then $\frac{A-b}{2}$ is the sum of the means, $\frac{A+b}{2}$ the sum of the extremes, and $\therefore$ the proportion sought is $\frac{1}{4}[(A+b-\sqrt{(2c-8a+2bA)}] : \frac{1}{4}[A-b-\sqrt{(2c-8a-2bA)}] = \frac{1}{4}[(A-b+\sqrt{(2c-8a-2bA)}] : \frac{1}{4}[A+b+\sqrt{(2c-8a+2bA)}]$.

When $a=18$, $b=2$, $c=130$, then $2 : 3 = 6 : 9$.

When $a=270$, $b=20$, $c=3922$, then $5 : 9 = 30 : 54$.

65. In a geometrical proportion the product of the two extremes or means $= a$, the sum of all the terms $= b$, and the difference between the sum of the squares of the extremes, and the sum of the squares of the means $= c$. What is the proportion?

Ans. The sum of the two means is $\frac{b^2-c}{2b}$, the sum of the two extremes is $\frac{b^2+c}{2b}$, and consequently the proportion sought is

$$\frac{b^2+c-\sqrt{[(b^2+c)^2-16ab^2]}}{4b} : \frac{b^2-c-\sqrt{[(b^2-c)^2-16ab^2]}}{4b}$$

$$= \frac{b^2-c+\sqrt{[(b^2-c)^2-16ab^2]}}{4b} : \frac{b^2+c+\sqrt{[(b^2+c)^2-16ab^2]}}{4b}$$

66. Required to find three numbers in continual proportion, whose sum $= a$, and the sum of whose squares $= b$?

Ans. The middle term of the required proportion is $\frac{a^2-b}{2a}$; the two extremes consequently are

$$\frac{a^2+b-\sqrt{(3b\ \ a^2)\cdot(3a^2-b)}}{4a}, \quad \frac{a^2+b+\sqrt{(3b-a^2)(3a^2-b)}}{4a}$$

Between what limits must the value of b fall, in order that the problem may give possible results?

67. In a continual proportion the sum of all three terms $= a$, and the remainder, which we obtain when we subtract the square of the means from the sum of the squares of the extremes $= b$. What proportion is it?

Ans. Call the middle term g; then $g = \frac{-a \pm \sqrt{(3a^2-2b)}}{2}$; hence we obtain the extremes

$\frac{1}{2}[a-g \pm \sqrt{(a^2-2ag-3g^2)}]$,
$\frac{1}{2}[a-g \mp \sqrt{(a^2-2ag-3g^2)}]$.

68. In a geometrical progression of four terms, the sum of all the terms $= a$, and the sum of their squares $= b$. What is the progression?

Ans. Let s denote half the sum, and d half the difference of the 2 means; then $s = \frac{-b \pm \sqrt{[b^2+2a^2(a^2-b)]}}{4a}$, and $d = s\sqrt{\frac{a-4s}{a+4s}}$; hence we obtain the 2 means $s-d$, $s+d$, and the 2 extremes $\frac{(s-d)^2}{s+d}$ and $\frac{(s+d)^2}{s-d}$.

69. In a geometrical progression of four terms the following things are given: the difference between the sum of two extremes, and the sum of the two means $= a$, and also the difference between the sum of the squares of the two extremes, and the sum of the squares of the two means $= b$. What is the progression?

Ans. Let s be half the sum of the means, d half their difference; then $s = \frac{b-a^2}{4a}$, $d = \frac{b-a^2}{4(2b-a^2)}$;

hence we obtain the means $s-d$, $s+d$, and the extremes $\frac{(s-d)^2}{s+d}$ and $\frac{(s+d)^2}{s-d}$.

70. In a geometrical progression of four terms we have the difference between the sum of the 2nd and 4th terms, and the sum of the 1st and 3rd terms $= a$, and also the sum of the squares of all four terms $= b$. What progression is it?

Ans. Let half the difference of the means $= d$, half their sum $= s$; then $d = \frac{b \pm \sqrt{[b^2+2a^2(a^2-b)]}}{4a}$, $s = d\sqrt{\frac{a+4d}{a-4d}}$. Hence we obtain, as in the two foregoing problems, the means and extremes.

71. In a geometrical progression of four terms we have given the sum of all the terms $= a$, the difference between the sum of the squares of the extremes and the sum of the squares of the means $= b$. Find the progression.

Ans. Half the sum of the means $s = \frac{a^2-b}{4a}$, half the difference $d = \pm s\sqrt{\frac{b}{8as+b}}$; whence we obtain the others, as in the three preceding problems.

72. In a geometrical progression of four terms we have the sum of the two extremes $= a$, the sum of the two means $= b$. What is the progression?

Ans. Let the ratio of the progression be e; then $e = \frac{a+b \pm \sqrt{(a-b)(a+3b)}}{2b}$, and the 1st term $= \frac{a}{e^3+1}$, $= \frac{b}{e^2+e}$.

73. In a geometrical progression we have the sum of the means $= a$, the sum of the extremes $= b$, the sum of the cubes of all four terms $= c$. What is the progression?

Ans. Let the product of the means or extremes be p; then $p = \frac{a^3+b^3-c}{3(a+b)}$, and the required proportion $\frac{1}{2}[b-\sqrt{(b^2-4p)}] : \frac{1}{2}[a-\sqrt{(a^2-4p)}] = \frac{1}{2}[a+\sqrt{(a^2-4p)}] : \frac{1}{2}[b+\sqrt{(b^2-4p)}]$.

74. In a geometrical proportion the sum of all the terms $= a$, the sum of their squares $= b$, the sum of their cubes $= c$. Required the proportion?

Ans. The product of the means or extremes $p = \frac{a^3-3ab+2c}{6a}$, the difference between the sum of the two extremes and the sum of the two means $d = \pm\sqrt{\frac{a^3-6ab+8c}{3a}}$; consequently the sum of the two extremes $= \frac{a+d}{2}$, and the sum of the two means $= \frac{a-d}{2}$; hence therefore the proportion is $\frac{1}{4}[a+d-\sqrt{[(a+d)^2-16p]}] : \frac{1}{4}[a-d-\sqrt{[(a-d)^2-16p]}] = \frac{1}{4}[a-d+\sqrt{[(a-d)^2-16p]}] : \frac{1}{4}[a+d+\sqrt{[(a+d)^2-16p]}]$.

75. In a geometrical proportion the difference between the sum of the extremes and the sum of the means $= a$, the difference between the sum of the squares of the extremes and the sum of the squares of the means $= b$; farther, the difference between the sum of the cubes of the

extremes and the sum of the cubes of the means $= c$. What proportion is it?

Ans. The sum of all the terms $= \frac{b}{a}$; hence the sum of the extremes $s = \frac{b+a^2}{2a}$, the sum of the means $s' = \frac{b-a^2}{2a}$, the product of the extremes or means $p = \frac{a^4+3b^2-4ac}{12a^2}$. If s, s', and p, be found, then the proportion sought is

$$\tfrac{1}{2}[s - \sqrt{(s^2-4p)}] : \tfrac{1}{2}[s' - \sqrt{(s'^2-4p)}]$$
$$= \tfrac{1}{2}[s' + \sqrt{(s'^2-4p)}] : \tfrac{1}{2}[s + \sqrt{(s^2-4p)}].$$

76. In a geometrical proportion the product of the two extremes or means $= a$, the sum of all the terms $= b$, and the sum of their cubes $= c$. Required the proportion.

Ans. For shortness-sake, let $\pm \frac{\sqrt{4c+12ab-b^3}}{3b} = A$; then is $\frac{b+A}{2}$ the sum of the extremes, & $\frac{b-A}{2}$ is the sum of the means; hence we obtain the proportion sought

$$\tfrac{1}{4}[b + A - \sqrt{[(b+A)^2-16a]}] : \tfrac{1}{4}[b - A - \sqrt{[(b-A)^2-16a]}] = \tfrac{1}{4}[b - A + \sqrt{[(b-A)^2-16a]}] : \tfrac{1}{4}[b + A - \sqrt{[b+A)^2-16a]}].$$

77. In a geometrical proportion the sum of the two extremes $= a$, the sum of the two means $= b$, and the sum of the cubes of all the four terms $= c$. What is the proportion?

Ans. For shortness-sake, let $\sqrt{\frac{4c-a^3-4b^3+3a^2b}{3(a+b)}}$

$= A$, $\sqrt{\frac{4c-4a^3-b^3+3ab^2}{3(a+b)}} = B$; then $\frac{a-A}{2}$:

$\frac{b-B}{b} = \frac{b+B}{2} : \frac{a+A}{2}$ is the proportion sought.

78. How are the two following equations solved, in which x', x'' are the quantities sought?

$(x'+x'')(1+x'x''+x'^2x''+x'x''^2+x'^2x''^2)+x'x''=a$,

$x'x''(x'+x'')(x'+x''+x'x'')(x'+x''+x'x''+x'^2x''+x'x''^2)$

$=b$.

Ans. If in these equations we successively make the substitutions $x'+x''=y'$, $x'x''=y''$, $y'+y''=z'$, $y'y''=z''$, $z'+z''=w'$, $z'z''=w''$, we obtain at length $w'+w''=a$, $w'w''=b$. The quantities sought, viz. x', x'', are therefore expressed by the four following quadratic equations:—

$$w^2-aw+b=0$$
$$z^2-w'z+w''=0$$
$$y^2-z'y+z''=0$$
$$x^2-y'x+y''=0$$

The 1st gives w', w''; the 2nd z', z''; the 3rd y', y''; and lastly the 4th x', x''. We thus successively obtain

$$w' = \frac{a \pm \sqrt{(a^2-4b)}}{2}, \quad w'' = \frac{a \mp \sqrt{(a^2-4b)}}{2}$$

$$z' = \frac{w' \pm \sqrt{(w'^2-4w'')}}{2}, \quad z'' = \frac{w' \mp \sqrt{(w'^2-4w'')}}{2}$$

$$y' = \frac{z' \pm \sqrt{(z'^2-4z'')}}{2}, \quad y'' = \frac{z' \mp \sqrt{(z'^2-4z'')}}{2}$$

$$x' = \frac{y' \pm \sqrt{(y'^2-4y'')}}{2}, \quad x'' = \frac{y' \mp \sqrt{(y'^2-4y'')}}{2}$$

and consequently for x' as well as for x'' 16 different values.

If we wish to solve the above equations by the common method, we shall, after a laborious elimination, arrive at an equation of the 16th degree.

XVIII. PROBLEMS FOR EQUATIONS OF HIGHER DEGREES.

1. What number is it, whose 3rd part multiplied by its 4th, gives the number 1944? *Ans.* 18.

2. What number is it, whose half, third, and fourth multiplied together, and the product increased by 32, gives 4640? *Ans.* 48.

3. It is required to find a number, such, that if its 4th power be divided by its 8th part, and 167 subtracted from the quotient, the remainder may be 12000. What number is it? *Ans.* $11\frac{1}{2}$.

4. Some merchants engage in business; each gives to it a thousand times as many pounds as there are partners. They gain in this business 2560*l.*; and it is found, after arranging the accounts, that they have gained exactly half their own number per cent. How many merchants are there? *Ans.* 8.

5. A capitalist puts 10000*l.* out to interest, and adds the interest yearly to the capital. At the end of the 3rd year he finds his capital increased to 11576*l.* 5*s.* What interest did it bear yearly? *Ans.* 5 per cent.

6. It is required to find three numbers of the following properties: if the square of the 1st be multiplied by

the 2nd, the product = 112; if the square of the 2nd be multiplied by the 3rd, we get 588; but if we multiply the square of the 3rd by the 1st, we obtain 576. What numbers are they? *Ans.* 4, 7, 12.

7. Find three numbers, such, that the square of the 1st multiplied by the 2nd = a; the square of the 2nd multiplied by the 3rd = b; and the square of the 3rd multiplied by the 1st = c. What numbers are they?

$$Ans.\ \sqrt[9]{\frac{a^4c}{b^2}},\ \sqrt[9]{\frac{b^4a}{c^2}},\ \sqrt[9]{\frac{c^4b}{a^2}}.$$

8. But when four numbers are sought, and it is required that the products a, b, c, d should be found successively when the square of each number is multiplied by the succeeding one, and the square of the last again is multiplied by the 1st; what numbers will they be then?

$$Ans.\ \sqrt[15]{\frac{a^8c^2}{b^4d}},\ \sqrt[15]{\frac{b^8d^2}{c^4a}},\ \sqrt[15]{\frac{c^8a^2}{d^4b}},\ \sqrt[15]{\frac{d^8b^2}{a^4c}}.$$

(Similar formulæ may also be found for five, six, or more numbers. The problem likewise admits of being made more general, all which is left for the reader's own consideration.)

9. A person bottled off a certain quantity of wine from a full wine-cask, which contained 81 quarts. When he had, afterwards, filled it again with water, he drew off again, from this mixture, exactly as much as before; this he did four times successively, until there were no more than 16 quarts of pure wine in the cask, the remainder being water. How many quarts did he draw off each time?

Ans. 27.

10. There are two numbers, whose difference is 4, and moreover, are such, that their product multiplied by their sum gives 1386. What numbers are they?

Ans. 7 and 11.

11. A person buys a silver vessel, which weighs exactly as many lbs. as there are ozs. of pure silver in each lb. He pays 120*l.* for this vessel; viz. for each oz. of pure silver which it contained 6*s.* 8*d.* more than the vessel would cost, if he paid 10*d.* for each lb. of its weight. How much does it weigh? *Ans.* 12 lb.

cubic. 12. Some officers were in the field with a detachment, partly infantry and partly cavalry. Each officer has under his command three times as many cavalry, and seven times as many foot as there are officers. Each cavalry soldier has 2, and each foot-soldier 22 cartridges more than there are officers; they had altogether 15360 cartridges. How many officers were there? *Ans.* 8.

13. A person was asked, how much he had expended that day—"To day," he answered, "I have spent 4*l.* more, and yesterday twice as much as I did the day before yesterday; if I multiply together the sums in pounds which I have expended in these three days, and add 756 to the product, I obtain exactly 134 times as much as I have expended to day." How much, therefore, is it?

Ans. 6*l.* or 9*l.*

14. Some merchants jointly form a certain capital, in such a way that each contributes ten times as many pounds as they are in number; they trade with this capital, and gain as many pounds per cent as exceed the number of

merchants by eight. Their profit amounts to 288*l.* How many were there of them? *Ans.* 12.

15. Some merchants collect a capital of 8240*l.* To this, each contributes forty times as many as there are of them. With this whole sum they gain as many pounds per cent as there are persons. They then divide the profit, and each takes ten times as many pounds as there are persons; but after this there remain 224*l.* over. How many merchants were there?

Ans. Either 7, or 8, or 10.

16. Four persons, A, B, C, D, have each of them a certain number of pounds in their possession, B 1*l.* more than A, C 1*l.* more than B, and D 1*l.* more than C. If we multiply the four sums by one another, and consider the product as so many pounds, we shall obtain 1168*l.* more than we should by cubing D's. How much has each?

Ans. A 5*l.*, B 6*l.*, C 7*l.*, D 8*l.*

17. A person has a certain number of workmen who are three times as many in number as each receives daily in ten-pences. They jointly work exactly 100 days less than their daily hire amounts to in ten-pences, and for this time they receive jointly 2500*l.* How many workmen were there? And how many days did they work?

Ans. 30 workmen, and 200 days.

18. I have two numbers whose sum is 63. If the greater be divided by the less, this quotient multiplied by the greater, and $20\frac{1}{4}$ added to the product, there arises a cubic number, whose root is 1 less than the 7th part of the greater number. What are the numbers?

Ans. 35, 28.

19. A reservoir is supplied with water by 4 pipes, and can be filled by them in $115\frac{1}{5}$ minutes. But if the reservoir be filled by each pipe singly, then the 2nd will require 4, the 3rd 8, and the 4th 12 hours more than the 1st. In what time will it be filled by the 1st?

Ans. In 4 hours.

20. Three numbers are given by the following data: the sum of the 1st and 2nd is $=a$; the sum of the squares of the 2nd and 3rd $=b$; and the sum of the cubes of the 1st and 3rd $=c$. By what equation will these 3 numbers be determined?

Ans. Let x denote the 1st of the three numbers sought, then is $\{b-(a-x^2)\}^3=(c-x^3)^2$, the equation which we have to solve. If x be determined from hence, then the other two numbers may easily be found. A problem may be considered as solved, so soon as we have reduced it to the most simple equation it admits of, although we are not always able to solve the equation itself completely.

21. We know that the sum of two numbers $=a$, that the sum of their sixth powers $=b$: how are these numbers found?

Ans. The product p of the two numbers is expressed by the equation $2p^3-9a^2p^2+6a^4p-a^6=0$.

22. The sum of two numbers $=a$, the sum of their seventh powers $=b$: how are these numbers found?

Ans. The product p of the two numbers is expressed by the equation $7ap^3-14a^2p^2+7a^3p-a^7+b=0$.

In general, the two equations $x+y=a$, $x^{2n}+y^{2n}=b$, or $x^{2n+1}+y^{2n+1}=b$, always lead to an equation of the nth degree for p, whose law may also be represented.

23. m times the difference of two numbers added to n times their product, gives a; their difference multiplied by the sum of their squares, gives b: what numbers are they?

Ans. Their difference $= y$, and their product $= z$, are expressed by the equations

$$ny^3-2my^2+2ay-nb=0,\ nz=a-my$$

Having found y and z from these, we can also determine the quantities themselves by the solution of a quadratic equation. The two quantities are correctly given by equations of the sixth degree, which, however, as on inspection may be seen, can be reduced to cubic and quadratic equations.

24. The sum of three numbers $= a$, the sum of their squares $= b$, the sum of their cubes $= c$. How are they found?

Ans. The sum of the products of 2 and 2 $= \frac{a^2-b}{2}$, and the product of all 3 $= \frac{2c+a^3-3ab}{6}$. Consequently the equation

$$x^3-ax^2+\frac{a^2-b}{2}x-\frac{2c+a^3-3ab}{6}=0$$

gives all three numbers.

25. The sum of the squares of two numbers $= a$, the sum of their cubes $= b$. What numbers are they?

Ans. Let s denote the sum of the two numbers,

then $s^3-3as+2b=0$ is the equation by which it may be found. Having determined s, the numbers themselves may be easily found; they are given by the equation $x^2-sx+\frac{s^2-a}{2}=0$.

26. The sum of the product of three numbers, two and two, $=a$, the sum of their squares $=b$, the product of all three $=c$. How are they found?

Ans. The sum of the three numbers $=\pm\sqrt{(2a+b)}$; consequently the equation, by which they are given at once, is $x^3\mp x^2\sqrt{(2a+b)}+ax-c=0$.

27. The sum of the products of three numbers, taken two and two, $=a$, the sum of their squares $=b$, the sum of their cubes $=c$. How are these numbers found?

Ans. The sum of the three numbers $=\pm\sqrt{(2a+b)}$, the product of all three $=\frac{1}{3}[c\pm(a-b)\sqrt{(2a+b)}]$. The following equation, therefore, gives all three numbers at once: $x^3\mp x^2\sqrt{(2a+b)}+ax-\frac{1}{3}[c\pm(a-b)\sqrt{(2a+b)}]$.

28. The sum of the products of three numbers, taken two and two, $=a$, the sum of their squares $=b$, the sum of the 6 products arising from multiplying each number by the square of another $=c$. How are they found?

Ans. The sum of the three numbers $=\pm\sqrt{(2a+b)}$, and their product $=\frac{1}{3}[\pm a\sqrt{(2a+b)}-c]$; they are, consequently, given by the following equation: $x^3\pm x^2\sqrt{(2a+b)}+ax-\frac{1}{3}[\pm a\sqrt{(2a+b)}-c]=0$.

29. Required to find three numbers in continual proportion, whose sum $= a$, and the sum of their cubes $= b$.

Ans. If y denote the middle term of the proportion, then $3y^3 - 3a^2y + a^3 - b = 0$ is the equation by which it may be determined. Let w be a root of this equation, then

$$x^2 - (a - w)x - \frac{b - w^3}{3(a - w)} + \frac{(a - w)^2}{3} = 0$$

is the equation, the roots of which are the two extremes.

When $a = 21$, $b = 1971$, $y^3 - 441y + 2430 = 0$. The three roots of this equation are 6, $-3 + \sqrt{414}$, $-3 - \sqrt{414}$. If w be assumed $= 6$, then $x^2 - 15x + 36 = 0$ is the equation, whose roots 3 and 12 give the two extremes. The continual proportion is $\therefore 3 : 6 : 12$.

30. In a progression of four terms we know that the difference of the two extremes $= a$, and the sum of the two means $= b$. What is the progression?

Ans. If the ratio of the progression be denoted by y, then $by^3 - ay^2 - ay - b = 0$ is the equation, by which the same may be determined. Having found this, then the 1st term $= \frac{b}{y^2 + y} = \frac{a}{y^3 - 1}$.

31. But what will the progression be, when the sum of the two means $= a$, and the sum of the squares of the two extremes $= b$?

Ans. The difference of the two means $= d$, and assuming $d^2 = y$, then $y^3 + (15a^2 - 2b)y^2 +$

$(15a^4+4a^2b)\,y+a^4\,(a^2-2b)=0$ is the equation, from which y may be obtained. If y be determined from this, and consequently d, we then obtain the ratio of the progression $=\frac{a+d}{a-d}$, and the 1st term $=\frac{(a-d)^2}{2(a+d)}$.

If we know previously, either from the nature of the problem, which leads to a given equation, or from any other cause, something of the relations of the roots of this equation, then, with few exceptions, we are enabled to reduce the given equation lower. The following problem will explain this in some degree.

32. Let $2m$ quantities be given by the equation $x^{2m}+ax^{2m-1}+bx^{2m-2}+\ldots.kx+l=0$. We know already, that the sum of every two of these quantities is the same. How, then, will this equation admit of being solved by another of lower dimensions?

Ans. The sum of every 2 roots of this equation $=-\frac{a}{m}$. By arranging it according to the powers of $x^2+\frac{ax}{m}$, and then substituting y for this expression, we obtain an equation of the mth degree for y. If from hence we can find y, we then also get x from the equation $x^2+\frac{ax}{m}=y$. The proof of this depends upon the analysis of the equations into simple factors, and may be easily derived from it. This is left to the reader's consideration.

To this class belongs, for instance, the equation $x^4-10x^3+18x^2+35x-12=0$. We put $x^2-5x=y$, and give the equation the form $(x^2-5x)^2-7(x^2-5x)-12=0$. The equations $y^2-7y-12=0$, $x^2-5x-y=0$, then give the 4 required values of x, viz.

$$x=\frac{5\pm\sqrt{(39+2\sqrt{97})}}{2},\ x=\frac{5\pm\sqrt{(39-2\sqrt{97})}}{2}$$

It must be previously assumed, that the equation $x^6-12x^5+42x^4-16x^3-79x^2-68x-18=0$ has 3 pairs of roots, whose sums are equal; then we put $x^2-4x=y$, and give to each equation the form $(x^2-4x)^3-6(x^2-4x)^2+17(x^2-4x)-18=0$. Then the equations $y^3-6y^2+17y-18=0$, $x^2-4x=y$, give the 6 roots sought. The 3 values of y are 2, $2+\sqrt{-5}$, $2-\sqrt{-5}$; consequently, x has the 6 following values:

$$2\pm\sqrt{6},\ 2\pm\sqrt{(6+\sqrt{-5})},\ 2\pm\sqrt{(6-\sqrt{-5})}$$

With the help of indeterminate coefficients it will always be easy to give the equation the desired form.

XIX. INDETERMINATE, OR DIOPHANTINE PROBLEMS.

If from the conditions of a problem we cannot obtain as many equations as there are unknown quantities, then it belongs to the class of Indeterminate Problems. Let there be m equations of the form $ax+a'x'+a''x''+a'''x'''+$ &c. $=k$, between the M unknown quantities x, x', &c. and let $M>m$; then, by eliminating $m-1$ of these unknown quantities, we shall obtain an equation of the same form,

in which, however, there will only be $M-m+1$ of these unknown quantities. In order, then, to solve such an equation, it is only necessary to express one of the unknown quantities by all the rest, and then to substitute for the others these last arbitrary numerical values. If, for instance, we had the equation $3x+5x'-7x''=17$, we should find $x''=\frac{3x+5x'-17}{7}$. We can assume x and x' as arbitrary, and from them determine x''.

But there are usually certain other conditions added which cannot be so properly expressed by equations, by which means the thing becomes much more difficult. As, for instance, when it is required, that x, x', &c. shall all be whole numbers; and when, besides this, it is desired that they shall all be positive, and so on. In the one case we can always assume, that the coefficients k, a, &c. are whole numbers; and in the other we can get rid of the fractions by multiplying by a factor.

The equation $ax+a'x'=1$, when it is made a condition, that x and x' shall be whole numbers, can be solved either by the method given in most elementary books, or by the help of Continued Fractions. The latter method depends upon the rules VI, VII, and is much easier than the former. Having by any method found $x=p$, $x'=q$, so that $ap+aq'=1$, we can then generally put $x=p+a'n$, $x'=q-an$, and then assume for n any arbitrary whole, positive or negative number.

The equation $ax+a'x'=k$ then gives $x=kp+a'n$, $x'=kq-an$. The equation $ax+a'x'+a''x''+a'''x'''+$ &c. $=k$, consequently, gives $x=(k-a''x''-a'''x'''-$&c.$)\times p+a'n$, $x'=(kc-a''x''-$&c.$)q-an$, where for n, x'',

&c. all possible whole, positive, or negative numbers may be assumed.

The above m equations, with M unknown quantities, are therefore reducible to the equation $ax+a'x'=1$, and this is always possible, unless a and a' have a common measure. The co-efficients a, a', &c., when the equation $ax+a'x'+a''x''+$ &c. $=k$ is possible, must contain at least 2 which have no common measure.

1. What numbers, divided by 3, leave the remainder 1, and divided by 5, the remainder 2?

Ans. 7, 22, 37, 52, 67, 82, and so on; in general, all numbers of the form $5n+7$.

2. What numbers, divided by 8, leave the remainder 5, and divided by 11, the remainder 4?

Ans. 37, 125, 213, 301, 389, and so on; in general, all numbers of the form $88n+37$.

3. What numbers are divisible by 9, and when divided by 14, leave 8 for a remainder?

Ans. 36, 162, 288, 414, 540, and so on; in general, all numbers of the form $126n+36$.

4. A country-woman brings a number of eggs to market, more than 100, but less than 200. She is undetermined whether she will sell them in fifteens or by the dozen, for in the 1st case she would have 4 over, in the 2nd 10. How many eggs has she? *Ans.* 154.

5. A boy plays with nuts, the number of which lies between 1 and 400, and wishes to make some small heaps of them. If he put 13 in each heap, he has 9 over; but

if he put 17 in each, he has 14 over. How many nuts are there? *Ans.* 269.

The two last problems belong to the indeterminate class.

6. What numbers, divided by 3, 7, and 10, leave the remainders 2, 3, and 9, respectively?

Ans. 59, 269, 479, 689, 899, and so on; in general, all numbers of the form $210n+59$.

7. What numbers, divided by 6, 12, and 15, leave the remainders 1, 1, 10?

Ans. 25, 85, 145, 205, 265, and so on; generally, all numbers of the form $60n+25$.

8. What numbers, divided by 5, 6, 7, 8, leave the remainders 3, 1, 0, 5?

Ans. 133, 973, 1813, 2653, 3493, and so on; in general, all numbers of the form $840n+133$.

9. What numbers, divided by 4, 6, 9, 15, leave the remainder 3 for the first 3 numbers, and for the fourth the remainder 12?

Ans. 147, 327, 507, 687, 867, and so on; generally, all numbers of the form $180n+147$.

10. A general was asked how strong his regiment was. He answered, "My regiment is not 2000 men strong; I can, however, draw it up 5, 6, and 7 deep, without having one over; but if I wished to make it 11 and 13 deep, in the first case I should have 9 men over, and in the second 8 too few." How many men did the regiment contain?

Ans. 1890 men.

11. A captain wished to march out his company, which was between 100 and 200 men strong. If he march them 2, 4, 8, and 10 men deep he has one over each time. But if he march them 6 or 12 deep, then he has 5 over each time. What number of men did his company contain? *Ans.* 161.

12. A fortunate gamester counted the sovereigns he had won twice successively, the first time having told them by threes, he had 2 over, the second time he counted by fives, and had one over. After this he played again, he lost 6 sovereigns, and then reckoned the money he had left by 7 and 11 at a time, and found that he had 3 over each time. How many sovereigns did he win the first time he played?

Ans. 86, or 1241, or 2396, and so on.

13. Required to find two numbers such, that if the first be multiplied by 17, and the second by 26, the first product is 7 greater than the second. What numbers are they?

Ans. 5 and 3, or 31 and 20, or 57 and 37, and so on.

14. In a foundry there were two kinds of cannon cast. Each cannon of the first sort weighs 16 cwt., and of the second kind 25 cwt.; and yet, for the second kind there were used 100 lbs. of metal less than for the first. How many cannons were there of each kind?

Ans. Of the first 11, and of the second 7, or of the first kind 36, and of the second 23, and so on.

15. Required to find three numbers such, that if the first be multiplied by 7, the second by 9, and the third by 11, the first product may be 1 less than the second, and 2 greater than the third?

Ans. 5, 4, 3; or 104, 81, 66; or 203, 158, 129, and so on.

16. A number of men, women, and children, form a party of pleasure. Each man spends 19, each woman 10, and each child 8 shillings. The men jointly expended 7 shillings more than the women, and 15 shillings more than the children. How many men, women, and children, were there?

Ans. 13 men, 24 women, and 29 children; or 53 men, 100 women, and 124 children, and so on.

17. Divide 142 into two such parts, that the one may be divisible by 9, and the other by 14. What parts are they? *Ans.* 72 and 70.

18. Divide the number 1591 into two such parts, that the one may be divisible by 23, and the other by 34. What parts are they?

Ans. 1081 and 510, or 299 and 1292.

19. Into what two parts must we divide the number 4890, that the first part, divided by 37, may leave the remainder 3, and the second, divided by 54, the remainder 6?

Ans. Into 780 and 4110, or into 2778 and 2112, or into 4776 and 114.

20. A bailiff buys horses and oxen for 1770*l.*, and pays 31*l.* for each horse, but 21*l.* for each ox. How many horses and oxen did he buy?

Ans. 9 and 71, or 30 and 40, or 51 and 9.

21. A person buys 124 head of cattle, viz. pigs, goats, and sheep, for 400*l.* Each pig costs 4*l.* 10*s.*, each goat

3l. 3*s.* 4*d.*, and each sheep 1*l.* 5*s.* How many were there each kind?

Ans. 17, 99, 8; or 40, 60, 24; or 63, 21, 40.

22. Divide 30 into three such parts, that if the first part be multiplied by 7, the second by 19, and the third by 38, the sum of the three products = 745. What parts are they? *Ans.* 6, 11, 13.

23. Divide 100 into three such parts, that if the first be multiplied by 17, the second by 11, the third by 3, and the 3 products added together, the sum = 880. What parts are they?

Ans. 2, 69, 29; or 6, 62, 32; 10, 55, 35, and so on, in all 10 different answers.

24. Required to find three whole numbers such, that if the first be multiplied by 5, the second by 13, the third by 18, the sum of the product = 997; but if the first be multiplied by 11, the second by 20, and the third by 37, the sum of the product = 1866. *Ans.* 16, 29, 30.

25. Required to find two numbers, whose sum and product are equal.

Ans. If x and y denote the two numbers sought, then

x is arbitrary, and $y = \frac{x}{x-1}$.

26. Required to find two numbers, whose sum is to their product as m to n.

Ans. If x and y denote the two numbers sought, then

x is arbitrary, and $y = \frac{nx}{mx-n}$.

27. Required to find two whole numbers, whose sum and product are together = 139.

Ans. 1 and 69, 3 and 34, 4 and 27, 6 and 19, 9 and 13.

28. Required to find two whole numbers, whose product exceeds twice their difference by 100.

Ans. 10 and 10, 14 and 8, 22 and 6, 30 and 5, 46 and 4, 94 and 3.

29. How can we divide the fraction $\frac{230}{77}$ into 2 other fractions, whose denominators are 7 and 11?

Ans. Into $\frac{5}{7}$ and $\frac{25}{11}$, or into $\frac{12}{7}$ and $\frac{14}{11}$, or into $\frac{19}{7}$ and $\frac{3}{11}$.

30. Required to find two numbers, whose squares together, give a square number.

Ans. If p and q denote two arbitrary numbers, then one of the numbers sought $= p^2 - q^2$, and the other $= 2pq$, viz. 3 and 4, 6 and 8, 5 and 12, and so on.

31. Let a and c denote two rational numbers: what rational numbers can be assumed for x and y, so that the formula $a^2x^2 + cy^2$ may be a complete square?

Ans. $x = cn^2 - m^2, y = 2amn$; then $a^2(cn^2 - m^2)^2 + c \times (2amn)^2 = a^2(cn^2 + m^2)^2$. Arbitrary rational numbers may be assumed for m and n when a and c are found, and hence the values of x and y will follow.

32. What value may be assigned to the indeterminate quantity x, that the formula a^2x^2+c may be a complete square?

Ans. $x=\frac{cn^2-m^2}{2amn}$; then $a^2\left(\frac{cn^2-m^2}{2amn}\right)^2+c=\left(\frac{cn^2+m^2}{2mn}\right)^2$.

33. If a, b, c denote three rational numbers; what rational numbers may be assumed for x and y, so that the formula $a^2x^2+bxy+cy^2$ may become a complete square?

Ans. $x=m^2-cn^2$, $y=bn^2-2amn$; then $a^2(m^2-cn^2)^2 + b(m^2-cn^2)(bn^2-2amn) + c(bn^2-2amn)^2 = (am^2-bmn+acn^2)^2$.

34. What value can be assumed for x, so that the formula a^2x^2+bx+c may be a square?

Ans. $x=\frac{m^2-cn^2}{bn^2-2amn}$; then $a^2\left(\frac{m^2-cn^2}{bn^2-2amn}\right)^2 + b\left(\frac{m^2-cn^2}{bn^2-2amn}\right)+c=\left(\frac{am^2-bmn+acn^2}{bn^2-2amn}\right)^2$.

35. What value can be given to x, in order that the formula ax^2+bx+c^2 may be made a complete square?

Ans. $x=\frac{bn^2-2cmn}{m^2-an^2}$; then $a\left(\frac{bn^2-2cmn}{m^2-an^2}\right)^2 + b\left(\frac{bn^2-2cmn}{m^2-an^2}\right)+c^2=\left(\frac{cm^2-bmn+acn^2}{m^2-an^2}\right)^2$.

36. Let $x=w$ be such a value of x, that by substituting the formula ax^2+bx+c may become a complete square. How can more of such values be found?

Ans. If we substitute $w+py$ for x in the given formula, then it will be transferred into another of the form fy^2+gy+h^2. This formula is a

complete square, if y be put $= \frac{gn^2-2hmn}{m^2-fn^2}$, consequently, the given formula itself, if x be put $= w + \frac{p(gn^2-2hmn)}{m^2-fn^2}$.

37. Let it be assumed, that the formula $ax^2+bxy+cy^2$ admits of being analyzed into the two rational factors $mx+ny$, $m'x+n'y$; that, consequently, b^2-4ac is a complete square. What numbers must be assumed for x, y, so that the formula $ax^2+bxy+cy^2$ may be a complete square?

Ans. $x = np^2-n'q^2$, $y = m'q^2-mp^2$; for then we have $mx+ny = (m'n-mn')q^2$, and $m'x+n'y = (m'n-mn')p^2$, &c. &c.

38. How is the equation $x^2 - Ay^2 = 1$ solved, when A, denotes any other arbitrary number than a square number; and it is required, that x and y should be whole numbers?

Ans. The solution of this important problem is contained in the rules already given. The values of x and y are no other than the numerator and denominator of the approximating values of Continued Fractions. For example, when $A=106$, $x=4005$, $y=389$; when $A=124$, $x=4620799$, $y=414960$; when $A=133$, $x=2588599$, $y=224460$. Those which are given here are moreover the least numbers which can be found. This method was 1st taught by Lagrange; it always safely leads to the object, and is much easier than Pell's method, which Euler gives.

39. When $x=m$, $y=n$, are known values of x and y in whole numbers, which solve, the equation $x^2-Ay^2=1$, in which A is a whole number: how can other integer values of x and y, in whole numbers, which will solve this equation, be found from it?

The irrational part vanishes in the solution,

When $p=0$, $x=1$, $y=0$.

When $p=1$, $x=m$, $y=n$, which are the values already known.

When $p=2$, $x=m^2+An^2$, $y=2mn$.

When $p=3$, $x=m^3+3Amn^2$, $y=3m^2n+An^3$.

When $p=4$, $x=m^4+6Am^2n^2+A^2n^4$, $y=4m^3n+4Amn^3$, and so on.

How can the problem, in order to make the formula $fx^2+gxy+hy^2$ a square (when thus it is generally possible), be reduced to the problem, in order to solve $x^2-Ay^2=1$ in whole numbers?

40. Required to find three numbers, such, that their sum, as well as the sum of every 2, may be a perfect square?

Ans. 41, 80, 320; or, 22, 42, $68\frac{1}{4}$; and innumerable others.

41. Required to find two numbers, such, that their difference = the difference of their cubes.

Ans. $\frac{5}{7}$ and $\frac{3}{7}$, $\frac{8}{13}$ and $\frac{7}{13}$, $\frac{16}{19}$ and $\frac{3}{19}$, and innumerable others.

42. What remainders can a square number leave, when it is divided by the numbers 2, 3, 4, 5, 6, &c.?

Ans. Divided by 2, 3, and 4, it leaves the remainders 0, 1; by 5, the remainders 0, 1, 4; by 6, the remainders 0, 1, 3, 4; by 7, the remainders 0, 1, 2, 4; by 8, the remainders 0, 1, 4; by 9, the remainders 0, 1, 4, 7; by 10, the remainders 0, 1, 4, 5, 6, 9, and so on.

Can the formula $3x^n+2$ ever be a perfect square, if whole numbers be assumed for x? Can the formula $14x^n+3$ be such a one?

43. What remainders will a cube number give, if it be divided by the numbers 7, 8, 9?

Ans. Divide by 7, the remainders 0, 1, 6; by 8, the remainders 0, 1, 3, 5, 7; by 9, the remainders 0, 1, 8.

Can the formula $8x^n+6$, $18x^n+7$ ever be a perfect cube, when whole numbers are assumed for x?

44. Find two numbers, such, that the sum of their squares may be a product of 2 factors, each of which again is a sum of 2 squares?

Ans. The equation $(mm'+nn')^2+(mn'-nm')^2 = (m^2+n^2)(m'^2+n'^2)$ answers this question, in which, for m, n, m', n', all possible, whole, or fractional numbers may be assumed.

45. Find four squares, such, that their sum may be a product of 2 factors, one of which is a sum of 3 squares, the other a sum of 2 squares?

Ans. The equation $(mm'+nn')^2+(mn'-nm')^2+(pm')^2+(pn')^2 = (m^2+n^2+p^2)(m'^2+n'^2)$ answers the conditions.

46. Find four squares, such, that their sum may be divided into 2 factors, each of which is a sum of 3 squares?

Ans. The equation $(mm' + nn' + pp')^2 + (mn' - nm')^2 + (mp' - pm')^2 + (np' - pn')^2 = (m^2 + n^2 + p^2)(m'^2 + n'^2 + p'^2)$ gives the answer to this question.

47. Find four squares, such, that their sum may be equal to two factors, each of which again is a sum of four squares?

Ans. The equation $(mm' + nn' + pp' + qq')^2 + (mn' - nm' + pq' - qp')^2 + (mp' - pm' + qn' - nq')^2 + (np' - p'n + mq' - qm')^2 = (m^2 + n^2 + p^2 + q^2) \times (m'^2 + n'^2 + p'^2 + q'^2)$ gives the answer.

48. What values can we assign to the indeterminate quantities x, y, that the formula $x^2 + Ay^2$ may be a product of two factors of the same form?

Ans. $x = mm' + Ann'$, $y = mn' - nm'$; then $(m'm + Ann')^2 + A(mn' - nm')^2 = (m^2 + An^2) \times (m'^2 + An'^2)$.

49. Let a, a', &c. be the remainders arising from the division of the numbers A, A', &c. by a number k, then these numbers may be represented by the formulæ $nk + a$, $n'k + a'$, &c. and the remainder of the product A, A', A'', &c. is consequently the same with that of the product a, a', a'', &c. How may the remainder of a higher power be briefly derived from hence when it is divided by a number?

Ans. Split the given power into lower ones, as factors, and seek the remainders from these; the

product of these remainders divided by k, then gives the remainder sought. Proceed, in like manner, with the 2nd, and so on to the 4th and 8th step (successively). For example, the remainder of 543^{113} divided by 257, or, which is the same, the remainder of 29^{113} is $=57$.

"Let p be any prime number, and A another, not divisible by p: then the power A^{p-1} divided by p, will always leave the remainder 1." How may this very important proposition in arithmetic be proved?

50. When p is a prime number, and m any very great number, how then, by means of this proposition, may the remainder of a power A^m be obtained still more briefly than in the preceding problem?

Ans. Let m divided by $p-1$ give the remainder r, then the remainder of A^m is the same as that of A^r and $r < p-1$.

51. In a number of the Berlin monthly Journal I once found some treatises on the immense magnitude of the number which is represented by the double the power $9^{9^9} = 9^{387420489}$. There was much said there on the subject; and this number exceeds in magnitude all conception, for it is, according to my calculation, expressible by no less than 369693100 digits, which may be easily found by logarithms. But what remainders will this vast number leave, when it is divided by the prime numbers 11, 13, 17, 19?

Ans. By 11, the remainder 5; by 13, the remainder 3; by 17, the remainder 9; and by 19, the remainder 16.

52. What remainder does the power $A^{\frac{p-1}{2}}$ leave when divided by p, when p is a prime number, and A is not divisible by p?

Ans. Either $+1$ or -1, consequently either 1 or $p-1$.

53. When p is a prime number, and $a, b, \ldots\ldots t, u$, are whole, positive, or negative numbers: how many whole values of x are there at most between 0 and p, which can divide the formula $ax^m + bx^{m-1} + cx^{m-2} + \&c. + tx + u$ by p?

Ans. At most m, unless all the co-efficients $a, b, \ldots\ldots u$, are at the same time divisible by p; in which case each value of x is sufficient. Why?

How may all other values, which possess, in like manner, this property be represented by a limited number of formulæ derived from the values of x which fall between 0 and p, which make the former formula divisible by p?

54. When for x, y, only whole rational numbers may be assumed, does not the formula $ax^2 + bxy + cy^2$, in which a, b, c, are given numbers, represent each whole number indiscriminately, but only a certain class of whole numbers, which suit this formula. But how can such a formula be transformed into another $a'x'^2 + b'x'y' + c'y'^2$, which possesses the property, that it represents all numbers, which can be exhibited by the other, and, inversely, that all numbers which this includes may be also comprehended by the other?

Ans. Take any two whole numbers m, n, but

which have no common measure; determine from hence two others m', n', which solve the indeterminate equation $mn' - pm' = \pm 1$; of these, as we know, there are innumerable others; then put $x = mx' + ny'$, $y = m'x' + n'y'$, and substitute these values in the formula $ax^2 + bxy + cy^2$, then this will be transformed into another $a'x'^2 + b'x'y' + c'y'^2$, which possesses the property required.

It is worthy of remark, that always $b^2 - 4ac = b'^2 - 4a'c'$. Why?—The quantity $b^2 - 4ac$ ought to be called Determinate, because upon it the nature of the formula depends. It is also at the same time that quantity, by whose sign it may be perceived, whether the formula $ax^2 + bxy + cy^2$ may be analyzed into two real factors $kx + ly$, $k'x + l'y$ or not.

The Diophantine Problems belong to that beautiful and interesting part of pure arithmetic, where the numbers are not considered in respect to their general relations, which they have as quantities, but in respect to the peculiar properties, by which they differ from one another. Whoever wishes to inform himself fully on the subject, may read the last chapter in Euler's Algebra, "The Theorie des Nombres of Legendre," and the "Disquisitiones Arithmeticæ" of Gauss.

In reading the two above-mentioned works of Gauss and Legendre, I consider one rule in particular as worthy of remark, that which is known by the name of the "Rule of Reciprocity," of which Gauss, besides the proof in his work, has also given another still more simple, which

Legendre has adopted in the second edition of his work. Whether it is to be found in any other book besides, I am not aware. It is as follows:—

If p and q be any two prime numbers, then the remainder arising from $q^{\frac{p-1}{2}}$ divided by p, is the same as that arising from the division of $p^{\frac{q-1}{2}}$ by q, viz. both $+1$, or both -1, if the prime numbers p, q, are not both of the form $4n+3$. If, on the contrary, they are both of the form $4n+3$, then the remainders are the reverse of each other; viz. one is $+1$, when the other is -1, and *vice versâ*.

The 58th problem is the basis of a very extensive Theory of Trinomial Factors, on which subject there is in the above two works a great deal said which is partly new.

XX. PROBLEMS INVOLVING PROGRESSIONS AND FIGURATES.

1. A gentleman hires a servant, and promises him for the 1st year only 30*l.* in wages, but for each following year 4*l.* 10*s.* more than for the preceding. How much will the servant receive the 17th year of his engagement, and how much for all the 17 years together?

Ans. 102*l.* and 1122*l.*

2. If 3500*l.* be lent out at 4 per cent interest, and

each year 300*l.* be added to it: how much will the interest amount to at the end of 24 years? *Ans.* 6672*l.*

3. A Traveller who wishes to be at the place of his destination in 19 days, expedites his journey so much, that he goes each day a quarter of a mile more than he did the preceding. If, then, he travel $14\frac{1}{2}$ miles the last day, how many miles did he go the 1st day? And how many miles did his whole journey amount to?

Ans. 10 miles, and $232\frac{3}{4}$ miles.

4. What is the difference of an arithmetical series of 22 terms, whose 1st term is 1, and last term 15?

Ans. $\frac{2}{3}$.

5. How many terms does an arithmetical progression consist of, whose difference is 3, first term 5, and last term 302? *Ans.* Of 100 terms.

6. A person who was asked about his salary, answered, "At present, I have 550*l.*; but when I first entered into office, I had no more than 100*l.*; but, on account of my industry, I received each year an addition of 30*l.* to my income." How long was he engaged?

Ans. 16 years.

7. A debtor being unable to pay his debt amounting to 12990*l.* at once, agrees with his creditor to discharge it by monthly instalments, viz. 600*l.* the 1st month, but each succeeding month 50*l.* more than the preceding one. In how many months will he have discharged his whole debt? And how much does he pay the last month?

Ans. In 14 months, and 1250*l.*

8. According to natural philosophy, each body which falls in vacuo, performs in the 1st second of its fall a space of about $16\frac{1}{12}$ feet, but in each succeeding second $32\frac{1}{6}$ feet more than in the immediately preceding one. If a body have fallen 20 seconds, how many feet will it fall in the last second? And how many in the whole time?

Ans. $627\frac{1}{4}$ feet, and 6600 feet.

9. How long, according to the conditions of the foregoing problem, must a body fall to accomplish a space of 4116 feet? *Ans.* 16 seconds.

10. In a company the conversation turning on domestic œconomy, a person said, "This year I have saved 78*l.*, I have also, already made a fortune of 1350*l.* by the following means: since I first entered into employment till now, I have saved each year of my salary 2*l.* more than in the preceding. How long was the person in office? And how much did he save the 1st year?

Ans. 25 years, and 30*l.*

11. A person was fined 800*l.* by the judge, which were to be paid by instalments, the 1st one was 20*l.*, but in each succeeding period something invariably greater than in the preceding, so that in the last period he paid 80*l.* In how many payments will the whole fine be discharged? And what was the common difference between each successive payment?

Ans. 16 instalments, and 4*l.*

12. In a foundery a person saw 15 rows of cannon-balls placed one above another, and asked a bombardier, who stood near him, how many balls there were in the

lowest row? "You may easily calculate that," answered the bombardier, "in all these rows together there are 4200 balls, and each row, from the first to the last, contains 20 balls less than the one immediately below it." How many balls ∴ were there in the lowest row?

Ans. 140.

13. A meteorologist observed this remarkable phenomenon, that, from the 8th to the 19th of June of a certain year, the thermometer rose half a degree daily, and that the arithmetic mean of these 12 different positions of the thermometer was $18\frac{1}{4}$ degrees. At what degree did it stand the 8th day? *Ans.* At the 16th degree.

14. A company of soldiers which had successfully stormed a fortress, was rewarded as follows: that soldier who had mounted the wall first received a certain sum of money, the second a little less, the third exactly so much less than the second, and so on. When the money was divided, two of the soldiers could not be present on account of their wounds; their share was consequently given to two of their comrades, these two put both their own money and that of their two comrades into the same purse, and, afterwards, when they came to divide it, they had forgotten what fell to each man's share. One of them had received 92*l.* for himself and his comrade, and only remembered, that he himself was the 2nd, and his comrade the 7th; the other had received 71*l.* for himself and his comrade, and knew that he himself was the 11th, but his comrade the 4th. How much did each of the four soldiers receive?

Ans. The 2nd 54*l.* 15*s.*, the 4th 47*l.* 15*s.*, the 7th 37*l.* 5*s.*, and the 11th 23*l.* 5*s.*

15. A person has eighteen numbers, which are in arithmetical progression. If the two middle ones be added together, their sum is $31\frac{1}{2}$; but if the 1st and last be multiplied together, their product is $85\frac{1}{2}$. What is the 1st number, and the common difference? *Ans.* 3 and $1\frac{1}{2}$.

16. Two persons wishing to have an interview, travel at the same time on the same road from two different places, A and B, which are 170 miles distant from one another. He who starts from B travels regularly 4 miles a day; but the other only 2 miles the 1st day, and each following day he goes without interruption half a mile more than the preceding one. Where will they meet?

Ans. 102 miles from A.

17. A person enjoys a yearly income of 500*l.*, none of which he spends, but from the day on which he received it annually, beginning from to-day, when he drew it for the 1st time, he immediately puts it out at 5 per cent interest. In how many years will he thus accumulate 6875*l.* at simple interest? *Ans.* In 10 years.

18. In arsenals the cannon balls are sometimes piled up in triangular pyramids, viz. in such a way, that the uppermost ball rests on the top of 3 balls, these 3 on 6, these 6 balls again on 10, and so on; or, in short, so that the number of balls in the different layers, calculated from the 1st, are represented by the triangular numbers 1, 3, 6, 10, 15, 21, 28, 36, and so on. How many balls, then, are there in the lowest layer of the Δ, if its side contain n balls? and how many balls are there in the whole pyramid?

Ans. $\frac{n(n+1)}{1 \cdot 2}$ in the lowest row, and $\frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$ in the whole pyramid.

Example. When $n = 30$, the lowest layer contains $\frac{30 \cdot 31}{1 \cdot 2} = 465$ balls, and in the whole pyramid there are $\frac{30 \cdot 31 \cdot 32}{1 \cdot 2 \cdot 3} = 4960$ balls.

19. But when the pyramid in the preceding problem is incomplete, and each side of the highest layer contains m balls. How many balls are there in this case in the pile?

Ans. $\frac{n \cdot (n+1)(n+2)}{1 \cdot 2 \cdot 3} - \frac{m \cdot (m+1)(m+2)}{1 \cdot 2 \cdot 3}$.

20. Sometimes, however, the balls are also piled in quadrilateral pyramids, so that all the layers form squares, and the balls in the different layers, beginning from the top, are represented by the square numbers 1, 4, 9, 16, 25, 36, &c. If, then, there be n balls in each side of the lowest layer, $\therefore$ there are n^2 balls at the bottom, how many balls are there in the complete pyramid, and how many are there in the incomplete pyramid, when the side of the highest layer contains m balls?

Ans. $\frac{n(n+1)(2n+1)}{1 \cdot 2 \cdot 3}$ in the complete pyramid, and $\frac{n(n+1)(2n+1)}{1 \cdot 2 \cdot 3} - \frac{m(m+1)(2m+1)}{1 \cdot 2 \cdot 3}$ in the incomplete one.

21. If there be many balls to pile up, it is customary to make the layers rectangular. In the upper row let there be then only m balls; these rest on 2 layers, each containing $m+1$ balls; this again on three layers, each of which contains $m+2$ balls, and so on; viz. in each succeeding layer a row more, and in each row one ball more. By this

arrangement the layers will then successively contain the following number of balls: m, $2(m+1)$, $3(m+2)$, and so on; $\therefore$ there are in the nth layer $n(m+n-1)$. How many balls then, are there in the whole pile of n rows?

$$Ans. \frac{n\cdot(n+1)(2n+3m-2)}{1\cdot 2\cdot 3}.$$

22. From the rectangular rows it is usual to form also another kind of piles, which, however, to retain their equilibrium, require, that on two sides they should lean on other piles, or be supported in any other way. Let there be, for instance, m balls in the top row; under this 2 rows, each containing $m-1$ balls; under these 3 rows, each containing $m-2$ balls, and so on; so that the balls in the different rows are represented by the following numbers: m, $2(m-1)$, $3(m-2)$, and so on; and, consequently, in the nth layer there are $n(m-n+1)$ balls. How many balls are there in such a pile consisting of n layers?

$$Ans. \frac{n.(n+1)(3m-2n+2)}{1\cdot 2\cdot 3}.$$

23. If a pile of this kind be only supported on one side, then each succeeding layer usually contains one row more, but the number of balls in each row remains the same. Then the number of balls in the different layers will be successively represented by the following numbers: m, $2m$, $3m$, and so on; $\therefore$ in the nth layer there are nm balls. How many balls are there in a pile of this kind?

$$Ans. \frac{mn(n+1)}{1\cdot 2}.$$

24. When the number of balls in a complete triangular pyramid $= s$, what equation shall we have to solve, in

order to determine from it the number of rows, or the number of balls, in the side of the lowest row?

Ans. The equation $n^3+3n^2+2n-6s=0$.

25. But what equation for a four-sided pyramid?

Ans. $2n^3+3n^2+n-6s=0$.

26. A king in India, named Scheran, wished, according to the Arabic author Aseyhad, that Lessa, the inventor of chess, should himself choose a reward. He requested the grains of wheat which arise when 1 is calculated for the 1st square of the board, 2 for the 2nd, 4 for the 3rd, and so on; reckoning for each of the 64 squares, twice as many grains as for the preceding. When it was calculated, to the astonishment of the king, it was found to be an enormous number. What was it?

Ans. 18446744073709551615; a sum, which, according to a moderate calculation, could not be obtained in the whole earth in 70 years and upwards, if all the land were employed in the cultivation of wheat.

27. A person sowed a peck of wheat, and used the whole crop for seed the following year, the produce of this 2nd year again for seed the 3rd year, and so on. If, in the 10th year, he sowed 1048576 pecks, by how many times must the seed have increased each harvest, supposing the increase to have been always the same?

Ans. Four times.

28. In a country during peace the population increased every year in the same proportion, and so fast, that in the space of four years, it be-

came from 10000 to 14641 souls. By what part did the population increase yearly?

Ans. By the 10th part.

28. It is required to insert 10 terms between 1 and 3, so that there may arise a geometrical progression of 12 terms. What is the 2nd term of this progression?

Ans. $3^{\frac{1}{11}} = 1,105$.

29. What exponent has a geometrical progression of 32 terms, whose 1st term is 5, and whose last term is 80? What is the sum of this progression? And what its 20th term?

Ans. The exponent is 1,0935; the sum 881,62; the 20th term 27,351.

30. There are seven numbers which form a geometrical progression, and which are such, that when the first six are added together, the sum is $157\frac{1}{2}$, but when the last six are added together, the sum is 315. What are the numbers?

Ans. $2\frac{1}{2}$, 5, 10, 20, 40, 80, 160.

31. In a geometrical progression of 5 terms, we know that the sum of the even terms $= a$, and the sum of the odd terms $= b$. What is the progression?

Ans. Let A, B, C, D, E, be the 5 terms of the required progression; then the middle term $\frac{-b \pm \sqrt{(b^2+4a^2)}}{2}$; hence, further, $A = \frac{(a - \sqrt{[a^2 - 4C^2]})^2}{4C}$, $B = \frac{a - \sqrt{(a-4C^2)}}{2}$, $D = \frac{a + \sqrt{(a^2-4C^2)}}{2}$, $E = \frac{[a + \sqrt{(a^2-4C^2)}]^2}{4C}$.

32. The sum of the even terms of a geometrical progression of four terms $= a$, the sum of the odd terms $= b$: what is the progression?

Ans. The common ratio of the progression $= \frac{a}{b}$, and the 1st term $= \frac{b^3}{a^2+b^2}$, by which it may be found.

33. The sum of the even terms of a geometrical progression of $2n$ terms $= a$, the sum of the odd terms $= b$: what is the progression?

Ans. The ratio $= \frac{a}{b}$, and the 1st term

$$= \frac{b^{2n-1}}{a^{2n-2}+a^{2n-4}b^2+\&c.+b^{2n-2}} = \frac{b^{2n-1}(a^2-b^2)}{a^{2n}-b^{2n}}.$$

34. In a geometrical progression of 6 terms, we know that the sum of the two middle terms $= a$, and the sum of the two extremes $= b$: how is it found?

Ans. Let p be the product of the two middle terms, then $p = \frac{a^2}{2}.\frac{5a \pm \sqrt{(4ab+5a^2)}}{5a-b}$. From p and a we obtain the two middle terms $\frac{1}{2}[a-\sqrt{(a^2-4p)}]$, $\frac{1}{2}[a+\sqrt{(a^2-4p)}]$, and hence again the required progression.

35. In a geometrical progression we know that the sum of the terms $= a$, the sum of their squares $= b$, and the sum of their cubes $= c$: how is it found?

Ans. The ratio of the progression is

$$= \frac{a^4+2ac-3b^2 \pm \sqrt{12a\,(ac-b^2)(a^3-c)}}{a^4+3b^2-4ac}, \text{ and}$$

the 1st term $= \frac{b(1+c)+a^2(1-c)}{2a}$. From the

ratio, the 1st term, and the sum of the terms, the number of the terms, and the progression itself may be found.

36. In a geometrical progression the sum of the terms $= a$, the sum of their squares $= b$, and the sum of their 4th powers $= c$: how is it found?

Ans. The ratio of the progression is $= \frac{a^4b - b^3 \pm 2a\sqrt{(c - a^2b)(b^3 - a^2c)}}{b^3 + a^4b - 2a^2c}$, the 1st term $= \frac{b(1+c) + a^2(1-c)}{2a}$. Hence the number of terms and the progression itself may be found.

37. The common difference of an arithmetical progression of four terms $= d$, the product of all its terms $= a$. How is the 1st term found?

Ans. Let the 1st term $= x$, the product of the two extremes $= y$; then $x^2 + 3dx = y$, and the unknown quantity y is found from the equation $y^2 + 2d^2y = a$. Hence we obtain the four following values of x:

$$x = -\tfrac{3}{2}d \pm \sqrt{[\tfrac{3}{4}d^2 + \sqrt{(a + d^4)}]},$$
$$x = -\tfrac{3}{2}d \pm \sqrt{[\tfrac{3}{4}d^2 - \sqrt{(a + d^4)}]}.$$

38. Let the difference of an arithmetical progression of six terms $= d$, the product of all its terms $= a$. How is the 1st term found?

Ans. Let the 1st term $= x$, the product of the two extremes $= y$; then $x^2 + 5dx = y$, and the unknown quantity y is found from the equation $y^3 + 10d^2y^2 + 24d^4y = a$.

In general, the problem, "To find an arithmetical progression of $2m$ terms with the given difference d, such that the product of all its terms may be equal to a given quantity a," may be reduced to the solution of an equation of the 2nd and the mth degree. Then it would only be necessary to put $x^2 + (2m - 1)\,dx = y$, and we should find for y an equation of the mth degree.

XXI. PROBLEMS RELATING TO INTEREST AND ANNUITIES.

1. A capital a stands at compound interest; the rate of interest $= p$. What will this capital amount to in n years? *Ans.* ap^n.

2. A capital of 5000*l.* stands at 4 per cent compound interest. What will it amount to in 40 years?
Ans. 24005.103*l.*

3. What will 3200*l.* amount to at 3 per cent in 80 years? *Ans.* 34050.84*l.*

4. In a certain province there are at present 20000 inhabitants. If the population increase a 15th part yearly, how many inhabitants will it contain 100 years hence?
Ans. 14489276 nearly.

5. What will a capital of 12000*l.* amount to in 10 years, if it bear 6 per cent, and the interest be paid half-yearly? *Ans.* About 21673*l.* 6*s.* 8*d.*

6. How long must 3600*l.* remain at 5 per cent compound interest, so that it may become as much as 5000*l.* at 4 per cent for 12 years? *Ans.* Nearly 16 years.

7. How long must a capital a remain at the interest p, to become as much as a capital a' at the interest p' for n years?

$$Ans. \frac{\log. a' + n \log. p' - \log. a}{\log. p} \text{ years.}$$

8. What is the amount of a capital which stands at 4 per cent, that 15 years hence it may be equal in value with 4500*l.* at 6 per cent for 9 years?

Ans. About 4221*l.*

9. What is the amount of a capital, which at the interest p for n years, is of equal value with a capital a' for n years at the interest p'?

Ans. $\log. a = \log. a' + n' \log. p' - n \log. p$. By means of this equation, first $\log. a$, and hence the capital a may be found.

10. On the other hand, what is the rate of interest that a capital a in n years may be equal to a capital a' in n' years at the interest p'?

$$Ans. \log. p = \frac{\log. a' + n' \log. p' - \log. a}{n}.$$

11. A debt of 7693*l.* paid 5 per cent interest. If, 5 years hence, 576*l.* be paid off, and after 8 years 498*l.* more. What will be the amount of the remaining debt 10 years hence, if compound interest be reckoned?

Ans. About 11696*l.*

12. What does the half-year's interest of a capital amount to, at 5 per cent per annum, reckoning compound interest? Farther, what is the amount of a quarter's interest, on the same supposition?

Ans. The half-year's interest is 2,4695, and the quarter's 1,2272 per cent nearly.

13. What is the amount of $\frac{1}{n}$th part of a year's interest at the rate p?

Ans. $100\ (\sqrt[n]{p} - 1)$ per cent.

14. How long must a capital stand at 4 per cent compound interest, that it may double itself? And how long when it triples itself?

Ans. It doubles itself in between 17 and 18 years, and triples between 28 and 29 years.

15. How long must it stand at p per cent, in order that it may be m times as great?

Ans. $\frac{\log.\ m}{\log.\ p}$ years.

16. An usurer lent a person 600*l.*, and drew up for the amount a bond for 800*l.* payable in 3 years, bearing no interest. What did he take per cent, if compound interest be taken into consideration?

Ans. Somewhat above 10 per cent.

17. If the capital a in n years be increased to A pounds: what is the rate of interest?

Ans. $\log.\ p = \frac{\log.\ A - \log.\ a}{n}$ (p denotes the rate of interest).

18. A person has to pay a sum of 3750*l.* six years hence. How much must he pay in cash for this sum, reckoning 4 per cent discount, and also compound interest? *Ans.* About 2963*l.* 13*s.* 4*d.*

19. What is the present value of a capital a, which is due n years hence, computed at p per cent? *Ans.* $\frac{a}{p^n}$.

20. What is the present value of a stock of turf which, 20 years hence, produces a sum of 500*l.*, so that the money laid out on it may bear 4 per cent? *Ans.* 228*l.*

21. A town contains 20000 souls, and we know that the population has regularly increased $\frac{3}{100}$ yearly: what was its population 10 years ago? *Ans.* 14882.

22. Three persons bade at an auction. The first offered 30000*l.* in cash; the second offered 33500*l.* in three years without interest; lastly, the third offered to pay 40000*l.* in seven years, also without interest. Which of these three offers is the greatest, calculating 5 per cent compound interest? And by how much does it exceed the two others in value?

Ans. The first is the greatest, and it exceeds the second by 1061*l.*, and the third by 1573*l.* nearly.

23. In how many years will the population of a place become 10 times as great as it is at present, if the yearly increase amount to three persons in a hundred?

Ans. In about 78 years.

24. A capital of 800*l.*, which was well laid out, increased in the space of 6 years to 3600*l.* What did this capital gain per cent? *Ans.* 28½ nearly.

25. A capital a is put out at the interest p, and at the expiration of each year the interest is added to the principal; but at the same time it increases or diminishes yearly by the same sum b. What will this capital amount to n years hence?

Ans. It is $= ap^n \pm \frac{b(p^n-1)}{p-1}$, where the upper sign obtains for the capital increased by b, and the lower the capital diminished by b.

26. A capital of 6000*l.* stands at 5 per cent compound interest, and each year, besides the interest added to the capital, it is increased by 500*l.* How much will the capital, by these means, become in 10 years?

Ans. About 16062*l.* 5*s.* 10*d.*

27. What will a capital of 3740*l.* amount to in 8 years, if at the end of each year 450*l.* be added to it, reckoning 4 per cent interest? *Ans.* About 9264*l.* 16*s.* 8*d.*

28. A debt of 15467*l.* bears 5 per cent interest; at the end of each year 600*l.* are deducted. What will be the amount of the remainder of the debt at the expiration of 10 years? *Ans.* About 17647*l.*

29. From a capital of 5000*l.*, standing at 5 per cent interest, a person takes away 400*l.* yearly. What will the remainder be 10 years hence? *Ans.* About 3113*l.*

30. A person is bound for seven years successively, to pay at the beginning of each year 4000*l.*, but is behind hand with the payment. What will he owe at the beginning of the seventh year, if the interest be calculated at 4 per cent? *Ans.* 31593*l.*

31. A person puts a capital of 30000*l.* out at 4 per cent, takes at the end of each year from the interest he received 800*l.* for his support, and adds the remainder to the capital. What will the amount of the capital be 15 years hence? *Ans.* 38009*l.* 16*s.* 8*d.* nearly.

32. A person has put his whole fortune of 100000*l.* out at 5 per cent interest, but is unable to pay his expenses with the interest of this capital, because he requires for this purpose 6000*l.* yearly. He is consequently obliged, at the end of each year, to take as much away from his capital as will, with the interest he receives, make up the 6000*l.* If he continue thus, in how many years will he become a beggar? *Ans.* In from 36 to 37 years.

33. A capital a is lent out at the rate p. In what time will it become a', if the capital, with the interest and compound interest added to it, be augmented or diminished yearly by the sum b?

Ans. $\text{Log. } n = \dfrac{\log[(p-1)a' \pm b] - \log[(p-1) \pm b]}{\log p}$

(n the number of years). The upper sign of $\pm$ denotes the addition of b, the lower the deduction b.

34. If in the preceding problem, b be yearly taken away, and b be greater than the interest of the capital a; in this case, in how many years will the capital be spent?

Ans. $\text{Log } n = \dfrac{\log b - \log[b-(p-1)a]}{\log p}$ gives the number of years required.

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35. What is the Present Value w of an annuity r at p per cent, which a person has to enjoy n years?

Ans. $w = \frac{(p^n - 1)r}{(p-1)p^n}$.

36. A person, who has to enjoy an annuity of 500*l.* for six years, wishes to sell it. How much cash can a person give him for this annuity, calculating $3\frac{1}{2}$ per cent?

Ans. About 2664*l.* 5*s.* 10*d.*

37. What is the Present Value of an annuity of 350*l.* assigned for 8 years, at 4 per cent interest?

Ans. Something above 2356*l.* 9*s.* 2*d.*

38. How much can a person give for an annuity of 400*l.*, which has to run 12 years, if the interest be reckoned at 3 per cent?

Ans. Something above 3981*l.* 12*s.* 4*d.*

39. If an annuity r having n years to run, be worth in cash w, what is its amount?

Ans. $r = \frac{(p-1)p^n w}{p^n - 1}$.

40. A debt due at this present time amounting to 1200*l.* is to be discharged in seven yearly and equal payments. What is the amount of these payments, if the interest be calculated at 4 per cent? *Ans.* 200*l.* nearly.

41. What is the amount of that annuity, the receipt of which, for 13 years, is considered as equal in value to 20000*l.* cash, computing the interest at 4 per cent?

Ans. 2003*l.* nearly.

42. A company borrowed of the government 20000*l.*, and for this sum mortgaged a forest, which yielded a clear annual income of 1500*l.* How long can the government

retain this forest for the capital advanced, if the interest be calculated at 5 per cent? *Ans.* 22 years nearly.

43. A person wishes to obtain an annuity of 2000*l.* for 34580*l.* For how many years can this annuity be granted him, computing the interest at 4 per cent?

Ans. For about 30 years.

44. How long has an annuity to run, if its Present Value be considered as equal to w pounds, the interest being 5 per cent? *Ans.* $\text{Log } n = \dfrac{\log r - \log [r-(p-1)w]}{\log p}$.

45. An annuity r is required for the given space of n' years, whose Present Value is equal to another annuity r for n years; if both be calculated at p per cent, what is the annuity? *Ans.* $r' = \dfrac{(p^n-1)p^{n'-n}}{p^n-1}r$.

46. But how great is the required annuity, if it be payable for m years; consequently, if computing after a lapse of m years, it be payable n' years after this period?

Ans. $r' = \dfrac{(p^n-1)p^{n'-n+m}}{p^{n'}-1}$.

47. What is the Present Value of an annuity r, which rises in a geometrical proportion, whose ratio is e, if it continue for n years, at the rate of p per cent?

Ans. $\dfrac{r(e^n-p^n)}{(e-p)p^n}$ or $\dfrac{r\left(\dfrac{e^n}{p^n}-1\right)}{e-p}$.

The latter form is more easily calculated by logarithms.

48. What is the Present Value of an annuity for n

years, if the increase be in the arithmetical progression r, $2r$, $3r$, $4r$, &c. so that at the end of the 1st year r be paid, at the end of the 2nd $2r$, and so on, calculating p per cent?

Ans. Let the Present Value of an annuity r for n years, which always remaining the same, $= w$, then the Present Value of an annuity increasing in the given arithmetical progression $=$

$$\frac{1}{p-1}\left(pw-\frac{nr}{p^n}\right).$$

The following observation is intended for the reader. The last problem may be very easily solved by help of geometrical progressions, and then no longer belongs to this place. It is, however, only a particular case of a much more general one. Let $A+Bx+Cx^2+$, &c. be the increasing annuity after the undetermined xth year; then the Present Value of this single increase $= p^{-x}(A+Bx+Cx^2+$ &c.), and consequently $Sp^{-x}(A+Bx+$ &c.) is the Present Value of the annuity, where S is the symbol of summation. Calculations of this kind are, however, always practicable. (See Traité du Calcul des Différences et des Séries par Lacroix, p. 90, § 911 of the first edition.) In the above problem $B=r$, and A, C, D, &c. $= 0$.

XXII. PROBLEMS RELATING TO PERMUTATIONS, COMBINATIONS, AND VARIATIONS.

1. How often can eight persons standing near one another change their places, that is, so change them, that they may present another order each time?

Ans. 40320 times.

2. How often can the 24 letters of the alphabet be transposed?

Ans. 620448401733239439360000 times. All the inhabitants of the globe, on a rough calculation, could not, in a thousand million of years, write out all the transpositions of the 24 letters, even supposing that each wrote 40 pages daily, each of which pages contained 40 different transpositions of the letters.

3. How many changes are there contained in all the permutations of *abcdefg*, which begin with one of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*? *Ans.* 720.

4. How many do they contain beginning with *ab*? How many with *abc*? How many with *abcd*?

Ans. 120 with *ab*, 24 with *abc*, and 6 with *abcd*.

5. How many do they contain, in which the letters *abcd* remain together in the same order in which they stand here? *Ans.* 24.

6. On the other hand, how many do they contain, in which the letters a, b, c, d, are placed without any regard to their order? *Ans.* 576.

7. How many changes are there contained in all the transpositions $a^3b^5c^4$, which begin with c^3? *Ans.* 504.

8. How many, which begin with a^2b^2c? *Ans.* 140.

9. How many are there, in which a occupies a fixed place, say the fourth? *Ans.* 6930.

10. What is the sum of all the figures in all the permutations of 12234 taken together? *Ans.* 720.

11. How many twos, threes, fours, fives, are there contained in 90 numbers?

Ans. 4005 twos, 117480 threes, 2555190 fours, and 43949263 fives.

12. How many twos, threes, fours, and fives, are there contained in 60 numbers?

Ans. 1770 twos, 34220 threes, 487635 fours, and 5461512 fives.

13. From a pack of piquet cards, consisting of 32 cards well shuffled, a person, blindfold, and without picking, draws 15 cards. In how many similar possible ways can the cards so drawn be arranged?

Ans. In 565722720 ways.

14. The product *abcd* may be divided in 3 different ways, into smaller products, each consisting of two factors, viz. into $ab \times cd$, &c.; the product *abcdef* admits of 15 such divisions, viz. $ab \times cd$, &c. In how many different

ways can the product *abcdefg*, &c., containing $2n$ factors, be split into such products of two factors?

Ans. In $\frac{(n+1).(n+2)....(2n-1)2n}{(1\cdot 2)^n}$.

15. In how many different ways can the product *abcdefg*, &c., compounded of $3n$ factors, a, b, c, d, e, f, g, &c., be split into products, each containing three of these factors? *Ans.* In $\frac{(n+1)(n+2)....(3n-1)3n}{(1.2.3)^n}$ ways.

16. In how many different ways can the product *abcdefg*, &c., composed of mn factors, a, b, c, d, e, f, g, &c., be analyzed into products each containing m factors?

Ans. in $\frac{(n+1).(n+2).....(mn-1)mn}{(1.2.3.....m)^n}$ ways.

17. Every whole number N is either itself a prime number, or it is a product of equal and different prime numbers. Let $\therefore$ a, b, c, &c. denote prime numbers, then, in general, it can be put $= a^m b^n c^p$, provided we consider m, n, p, &c. as representing all possible whole numbers, in which is comprised 0, or zero. How can all the possible numbers be found, by which the number N is divisible? And how many divisors will it generally have?

Ans. For shortness-sake, let $1+a+a^2+.....+a^m=A$, $1+b+b^2+...b^n=B$, $1+c+$&c. $=C$, &c. then the terms of the product $ABCD$, &c. give all the divisors of the number N, and their number $=(m+1)\times(n+1)(p+1)$ &c. For instance, 360 being $= 2^3.3^2.5^1$, the product 4.3.2 gives the number of divisors of 360, and the product $(1+2+4+8)(1+3+9)\times(1+5)$ gives these divisors themselves, viz. 1, 2, 4, 8, 3, 6, 12, 24, 9, 18, 36, 72, 5, 10, 20, 40, 15, 30, 60, 120, 45, 90, 180, 360.

18. How many numbers may be found, which exactly have the given number of divisors n?

Ans. Let a, b, c, &c. be any prime numbers. If n be a prime number, then a^{n-1} is such a number as is required. If n be a compound number, analyze it into its factors m', m'', m''', &c. (it matters not whether they be simple or not), and then $a^{m'-1}b^{m''-1}c^{m'''-1}$, &c. will be the number required.

19. A complete polynomial of the Nth dimension for n quantities x, y, z, &c. means that which contains all the combinations of these n quantities from the 0th to the nth class, inclusively. Thus, for instance, a polynomial of the 4th dimension for two quantities is this: $a + bx + b'y + cx^2 + c'xy + c''y^2 + dx^3 + d'x^2y + d''xy^2 + d'''y^3 +$ &c. Of how many terms does a complete polynomial of the Nth dimension for n quantities consist of?

Ans. Of $\dfrac{(N+1)(N+2)\ \ldots\ (N+n)}{1\ .\ 2\ \ \ldots\ \ n}$ terms.

20. How many of these terms remain over, when all those which are divisible by x^p are excluded?

Ans. $$\frac{(N+1)(N+2)\ \ldots\ (N+n)}{1\ .\ 2\ \ \ldots\ldots\ \ n} - \frac{(N-p+1)(N-p+2)\ \ldots\ (N-p+n)}{1\ .\ 2\ \ \ldots\ldots\ \ n}.$$

21. How many are there over, when all those which are divisible by x^p and y^q are excluded?

$$Ans.\ \frac{(N+1)(N+2)(N+3)\ldots\ldots(N+m)}{1\ .\ 2\ .\ 3\ \ldots\ldots\ n}$$

$$-\frac{(N-p+1)(N-p+2)(N-p+3)\ldots(N-p+n)}{1\ .\ 2\ .\ 3\ \ldots\ldots\ n}$$

$$-\frac{(N-q+1)(N-q+2)(N-q+3)\ \ldots\ (N-q+n)}{1\ .\ 2\ .\ 3\ \ldots\ldots\ n}$$

$$+\frac{(N-p-q+1)(N-p-q+2)\ldots(N-p-q+n)}{1\ .\ 2\ \ldots\ldots\ n}.$$

22. Let it be assumed, that in n several events, which may happen, there are m cases, which are favourable to any hope or expectation founded upon them, $\therefore$ $n-m$ cases where an unfavourable result may be expected: what probability is there for a favourable, and what for an unfavourable event?

Ans. $\frac{m}{n}$ for the favourable, and $\frac{n-m}{n}$ for the unfavourable event.

What do these expressions mean, when $m = 0$? And what, when $m = n$?

23. In a wheel of fortune containing 13 different numbers, what proportion does the probability of drawing 6 numbers already agreed upon, bear to the probability of drawing from a pack of piquet-cards (32 in number), 8 numbers previously fixed upon, on the supposition, that the numbers as well as the cards are mixed together, and that the drawing takes place blind-folded? And in the case, when in these two games, the prizes, after the fixed numbers are drawn, are the same in number: what proportion must the stakes bear, when in reference to the pro-

bability of winning, both games may be considered as the same?

Ans. The probability of winning in the first game is to the probability of winning in the second game, as 11 to 67425; and the stakes in these two games are, in like manner, as 11 to 67425.

24. In how many ways may 40 different balls be arranged into two heaps, so that one may contain 33 balls and the other 7? *Ans.* In 18643560 ways.

25. In how many ways can 21 different balls be arranged into three heaps, each containing 3, 7, and 11 balls? *Ans.* In 42325920 ways.

26. In how many ways may 19 different balls be arranged into four heaps, each containing 2, 4, 5, and 8 balls? *Ans.* In 523783260 ways.

27. From a pack of piquet-cards (32 in number) in how many different ways can first 12 cards, and then 9 from the remaining 20, be taken? Or, which is the same, in how many ways can 32 cards be arranged in three parts, so that the first may consist of 12, the second 9, and the third 11 cards?

Ans. In 37924165406400 ways.

28. In the game of piquet, each of the two players receives 12 cards, and the other 8 are laid aside as a stock: how many different games is it possible to have in the distribution of the cards? *Ans.* 28443124054800.

29. In how many ways can the whole 52 cards be distributed amongst four whist players? Or, which is

the same, in how many ways can these 52 cards be arranged in four equal parts, each of which contains 13 cards?

Ans. In 8565126197851151797861440000 ways.

30. From a piquet-pack of cards well shuffled together, 9 cards are to be drawn blind-folded: what is the probability, that there are five of a certain colour, say clubs, amongst them? *Ans.* $\frac{12597}{584350}$ or about $\frac{1}{47}$.

31. What is the probability, that two events may happen together, one of which having the probability $\frac{m}{n}$, the other the probability $\frac{m'}{n'}$?

Ans. $\frac{m}{n} \cdot \frac{m'}{n'} = \frac{mm'}{nn'}$. Why?

32. What is the probability of three events happening together, when they singly have the probabilities $\frac{m}{n}$, $\frac{m'}{n'}$, $\frac{m''}{n''}$ respectively? *Ans.* $\frac{mm'm''}{nn'n''}$.

33. In a chest there are 24 balls, viz. six white, eight black, and ten red; at first there are 7 balls drawn blind-fold, and then again, from the remaining 17, three more. I bet that all the seven balls are red, and the three balls are all black. What is my chance of winning?

Ans. $\frac{5}{490314}$, and consequently there is little chance.

34. How many different throws can be made with 2, 3, 4, and, in general, with n dice?

Ans. With two dice 36, with three 216, with four 1296, and, generally, with n dice 6^n throws.

35. What is the probability with four dice of throwing the same number on all four? *Ans.* $\frac{1}{216}$.

36. What is the probability with four dice of throwing exactly two numbers alike, so that the remaining dice may have unlike numbers? What is it on the same conditions for 5, 6, and 7 dice? Lastly, what is it for 8 dice?

Ans. For four dice the probability sought $= \frac{5}{9}$; for five dice $= \frac{25}{54}$; for six dice $= \frac{25}{108}$; for seven dice $= \frac{35}{648}$; and for eight dice $= 0$, or the case is impossible.

37. What is the probability with 4 dice of throwing a triplet, or of throwing three numbers of the same kind, and no more? *Ans.* $\frac{3}{54}$.

38. What is the probability, with 5 dice, of throwing three numbers alike and no more, but in such a way, that the other dice have different numbers? What is it, on the same conditions, for 6, 7, and 8 dice? Lastly, what is it for 9 dice?

Ans. For five dice the required probability $= \frac{25}{162}$; for six, in like manner, it is $= \frac{25}{162}$; for seven $= \frac{175}{1944}$; for eight $= \frac{35}{1458}$; for nine $= 0$, or the case is impossible.

39. Which is greater: the probability of throwing two sixes with three dice, or from a shuffled pack of piquet-cards, of drawing three cards of the same colour?

Ans. The first is the more probable, and the probability of the first case is to that of the second as 755 to 504.

40. There were two chances offered me to bet upon. In one eight similar numbers and no more were to be thrown from 12 dice, but, at the same time, in such a way, that the remaining were all different numbers. In the other, I am to draw from a whole pack of cards (52 in number) five cards of the same colour. What game, if I were disposed to bet, ought I to give the preference to? And what proportion do the chances of winning at these games bear to one another?

Ans. The second chance is better than the first, and the probability of winning at it, is to that of winning at the first, as 1259712 to 104125, or about $12\frac{1}{10}$ to 1.

XXIII. MISCELLANEOUS PROBLEMS.

1. From a pack, consisting of 32 cards, three cards were drawn; upon each of these there were as many cards laid as were requisite, that their number, together with the number of pips in the cards on which they were placed, might make 15; but then there still remained eight cards over. What was the number of pips in the three first cards? *Ans.* 24.

2. From a pack containing a cards n cards were drawn; upon each of these were so many cards placed, that their number, and the number of pips of each lowest card, were together equal to s. There then remained r cards over. What was the number of pips in the n first cards?

Ans. $nc+n+r-a$.

3. A person buys a piece of cloth, and pays 7*l.* for every five ells; he then sells every 11 ells for 16*l.*, and gains by this bargain 24*l.* How many ells did the piece contain? *Ans.* 440.

4. Two travellers began their journeys, *A* with 100*l.* in his possession, *B* with 48*l.* On the road they were attacked by robbers, who took a part of their money from them, by which means *A* lost twice as much money as *B*, but still had three times as much money as the latter. How much was taken from each?

Ans. From *A* 88*l.*, and from *B* 44*l.*

5. A book-binder sold me two bound paper-books, the one which contained 48 sheets, for 14*s.*, the other, which contained 78 sheets, for 19*s.*; the binding and paper were the same in both. What was the price of the binding?

Ans. 6*s.*

6. A sum of 156*l.* was to be divided amongst 16 poor children, in proportion to their ages, in such a way, that each of the elder ones received exactly twice as much as the next younger. If, therefore, the youngest, according to this division, received 6*l.*, how much more did each preceding one receive? And how much the eldest?

Ans. 10*s.*, and 13*l.* 10*s.*

7. A debt of 2365*l.* is to be paid off in 34 instalments, so that in each instalment there are 3*l.* more paid than in the immediately preceding one. How much must be paid in the first instalment? *Ans.* 20*l.*

8. A cistern, which is filled by two pipes in the space

of 12 minutes, could be filled by one of these pipes alone in 20 minutes. In what time could it be done by the other pipe? *Ans.* In 30 minutes.

9. Required to find a number, such, that when it is added to 15, 27, and 45, there arise three numbers, which are in geometrical proportion? *Ans.* 9.

10. I have an arithmetical and a geometrical progression, each of three terms, and the sum of all the six terms = 96. The first term of the arithmetical progression is contained twice in the first term of the geometrical progression; the second term of the arithmetical is contained thrice in the second term of the geometrical progression, and the third term of the arithmetical is contained six times in the third term of the geometrical progression. What are the progressions?

Ans. 3, 6, 9, and 6, 18, 54.

11. *A*, *B*, *C*, wished to buy a picture, but neither of them had money enough for the purpose. *A* begged of *B* and *C* the half of their money, in order to enable him to buy it; on the other hand, *B* asked *A* and *C* only for the third part of their money, because he then would be enabled to buy it himself. On which *C* said to *A* and *B* lend me the fourth part of your money, and then I can buy it. How much money had each, and how much did the picture cost, supposing we know that the three persons had only sovereigns about them?

Ans. The picture cost either 17*l.*, and then *A* had 5, *B* 11, and *C* 13; or the picture cost 34*l.*, and then *A* had 10, *B* 22, and *C* 26, &c. &c.

12. Five friends, *A*, *B*, *C*, *D*, *E*, jointly spent a certain sum at an inn; this sum is to be paid by one of them, but on counting the sovereigns they had in their pockets (for none of them had smaller coin), no one had enough to pay it alone. If one pay it alone, the others must add a part of their money, so that *A* must contribute the fourth, *B* the fifth, *C* the sixth, *D* the seventh, and *E* the eighth part, of the others' money. How much did they spend? And how much had each?

Ans. They spent at least 879*l.*, and then *A* had 319*l.*, *B* 459*l.*, *C* 543*l.*, *D* 599*l.* and *E* 639*l.*, &c. &c.

13. Some persons, who were strangers to each other, agreed to travel in company, sharing the expenses equally, hired a coach for 342*l.* On the road three went from the agreement, and then each of the others was obliged to pay 19*l.* more than otherwise would have fallen to his share. How many persons were there at first?

Ans. 9.

14. An ingot of gold was sold at a loss for 420*l.*; if it had been sold for 570*l.*, then the gain would have been exactly four times as much as the loss is at present. What did it cost? *Ans.* 450*l.*

15. A man left his widow with child, and a clear property of 9000*l.* He ordered in his will, that in case his widow should have a son, he should receive three times as much as his mother; if, however, she had a daughter, then the latter was to receive only half as much as her mother. A short time after the man's death the woman

was brought to bed of twins, viz. of a daughter and a son. In this case, how must the property be divided?

Ans. The mother receives 2000*l.*, the son 6000*l.*, and the daughter 1000*l.*

16. A traveller starts from a certain place, and goes one mile the first day, two the second, three the third, four the fourth, and so on in progression. Five days after, another traveller starts from the same place, takes the same road, and goes 12 miles daily. On what day after the departure of the first will these travellers be together?

Ans. On the 8th day, and if they continue their journey at the above rate, they will meet again on the 15th day.

17. The first traveller in the preceding problem goes a miles the first day, but each following one d miles more than he does the preceding. The second starts n days after the first, and goes b miles daily. In what time will they come together?

Ans. In $\dfrac{-(2a-2b-d)\pm\sqrt{[(2a-2b-d)^2-8bdn]}}{2d}$ days.

18. Required to find two numbers, such, that their product is equal to their sum, and their sum, added to the sum of their squares, is equal to $15\frac{3}{4}$? *Ans.* 3 and $\frac{3}{2}$.

19. A shepherd was asked how many sheep he had. He answered, "If I count my sheep by fours, sixes, or nines, I have three over each time; if I count them by sevens or thirteens, I have one over each time,

but if I reckon them by elevens, the remainder is 7." How many sheep then had he? *Ans.* 183, or 36219, or &c.

20. Two artillery-men together filled 1000 cartridges, and used the same quantity of powder. One says to the other, "If I had filled as many cartridges as you have, I should have made use of 18 cwt. of powder." Then the other answers him, "Had I filled as many cartridges as you have, I should have used only 8 cwt. of powder." How many cartridges did each fill? And how much powder did each use?

Ans. The first filled 400, the second 600 cartridges, and each used 12 cwt. of powder.

21. How long must a pound remain at 5 per cent compound interest, so that it may become 1000*l.*

Ans. Between 141 and 142 years.

22. Divide the number 70 into three such parts, that the first multiplied by 7, the second by 8, and the third by 9, and then these three products added together, the sum may be equal to 561. What are the parts?

Ans. 1, 67, 2; or 2, 65, 3; or 3, 63, 4, and so on.

23. A schoolmaster gave his pupil two numbers to multiply, one of which was greater than the other by 75. When the scholar had finished the multiplication he proved it, and the product divided by the least factor, gave the quotient 227, and remainder 113. The school-master then found that it was multiplied wrong, and ordered the error to be corrected. When the pupil had found out the error, he said, that he had calculated only one too little in

the multiplication; no, said the master, not one, but 1000. What numbers were given to multiply?

Ans. 159 and 234.

24. In a solution of salt and water, the weight of the fresh water $= a$, the weight of the salt $= b$, therefore its contents $= \frac{b}{a+b}$. How much water must be added to it, that its contents $= g$?

Ans. $\frac{b}{g} - (a+b)$. The unit of the weight the same as that in which a and b are given.

What does this expression imply when $\frac{b}{g} < a+b$, consequently $\frac{b}{a+b} < g$?

25. Any substance A is p times as heavy as water; another matter is B only p' times as heavy as water: how much of the second matter must be combined with a quantity of the first matter, the weight of which $= g$, so that the mass thus compounded may be p'' times as heavy as water, supposing that p'' falls between p and p'?

Ans. $\frac{gp'(p-p'')}{p(p''-p')}$.

26. Lead is 11.324 heavier than water, but the light cork-wood is only 0.24 times, the heavier fir-wood is 0.45 times as heavy as water. How much cork must be united to a piece of lead of 60lbs., so that the composition may weigh exactly as much as a piece of fir of the same size, and consequently may float?

Ans. 65.846 ..... lbs.

27. Let w and x be two quantities, which depend upon one another in such a way, that when the n determinate values a, a', &c. are assigned to the quantity x, the quantity w has successively the n corresponding determinate values w, w', &c. How can w be so represented by a polynomial of x, that the given conditions may be fulfilled?

Ans. Let $w = A + Bx +$ &c. and give n terms to the second part of this equation, then substitute successively for w and x, the corresponding values w, a; w', a', &c.; we then obtain n equations, by which the co-efficients A, B, &c. may be found.

28. Let w, x, y be three quantities, which depend upon one another in such a way, that they simultaneously have the values w, a, b; w', a', b', &c. respectively. How can w be expressed by a polynomial of x and y?

Ans. Let $w = a + Bx + B'y + Cx^2 + C''Bey + C''y^2 + Dx^3 + D'x^2y +$ &c., and give the second part as many terms as there are corresponding values; then substitute for w, x, y, respectively, and successively, the values w, a, b; w', a', b', &c.; then we obtain exactly as many equations as are necessary to determine the co-efficients A, &c.

29. Let w, x, y, z, be four quantities, which depend upon each other in such a way, that they simultaneously have the values w, a, b, c; w', a', b', c', &c. How can w be expressed by a polynomial of x, y, z?

Ans. Let $w = A + Bx + B'y + B''z + Cx^2 + C'xy + C''xz, + C'''y^2 + C''''yz + C'''''z^2 +$ &c., and then proceed as in the two foregoing problems.

In the same manner we proceed, when more quantities and more corresponding values are given. These problems, moreover, both in Physics as well as in all parts of mathematics, are of the greatest service. They are also the bases of many analytical methods.

30. Find a number, such, that if it be multiplied by another number less than itself by 1, the product = 1.

Ans. The number is given by the equation $x^3 - 2x^2 + x - 1 = 0$; the only real root of this equation is 1.7548 . . . . .

31. What is the value of the Continued Fraction

$$\cfrac{1}{q + \cfrac{1}{q + \cfrac{1}{q + \&c.}}}$$

ad infinitum, when it has no other quotient but q?

Ans. $\dfrac{-q + \sqrt{(q^2+4)}}{2}$.

32. What is the value of the Periodic-Continued Fraction

$$\cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{q + \cfrac{1}{r + \&c.}}}}$$

ad infinitum, in which the quotients q, r are constantly repeated?

Ans. $-\frac{r}{2}+\sqrt{\left(\frac{r^2}{4}+\frac{r}{q}\right)}$.

33. But what is the value of a Continued Fraction the same as the preceding, when, instead of two quotients q, r, three quotients q, r, s, are repeated *ad infinitum*?

Ans. The value is

$$\frac{qrs+q+s+r-\sqrt{[(qrs+q+s+r)^2+4]}}{2(qr+1)}.$$

34. Again, what is the value of such a fraction, in which the quotients q, r, s, t, are repeated *ad infinitum*?

Ans. $$-\frac{qrst+qr+qt+st-rs}{2(qrs+q+s)}+$$
$$\frac{\sqrt{[(qrst+qr+qt+st+rs+2)^2-4]}}{2(qrs+q+s)}.$$

How can the law of this value be represented when more than four quotients are repeated periodically?

35. In a geometrical proportion, the sum of the two middle terms $=a$, the sum of the two extremes $=b$, and the sum of the 4th powers of all the 4 terms $=c$. What is the proportion?

Ans. Let d be the difference of the two middle terms then $d=\sqrt{[-a^2-2b^2\pm 2\sqrt{(c+2a^2b^2)}]}$, and the required proportion is

$\frac{1}{2}[b-\sqrt{(b^2-a^2+d^2)}]\cdot\frac{1}{2}(a-d)=\frac{1}{2}(a+d:$
$\frac{1}{2}[b+\sqrt{(b^2-a^2+d^2)}]$.

36. In a geometrical proportion the sum of all the terms $=a$, the sum of their squares $=b$, and the sum of their 4th powers $=c$. What is the proportion?

Ans. Let p denote the product of the two extremes, and consequently of the two means, and d the difference between the sum of the two extremes and the sum of the two means; then $p = \frac{a^2}{2} \pm \sqrt{\frac{a^4 + 2a^2b + b^2 - 2c}{8}}$, and $d = \sqrt{(2b + 8p - a^2)}$. The four terms of the required proportion are consequently

$$\tfrac{1}{4}\{a-d-\sqrt{[(a-d)^2-16p]}\}$$
$$\tfrac{1}{4}\{a+d-\sqrt{[(a+d)^2-16p]}\}$$
$$\tfrac{1}{4}\{a+d+\sqrt{[(a+d)^2-16p]}\}$$
$$\tfrac{1}{4}\{a-d+\sqrt{[(a-d)^2-16p]}\}$$

37. A Debtor is obliged to pay the sums a, a', a'' a''', &c., in the times n, n', n'', n''', &c., but wishes to pay his whole debt $a+a'+$&c. at once: when ought this to take place, if the rate of interest $= p$, and compound interest be given?

Ans. Let w, w', &c. be the Present Values of the different payments, so that $w = \frac{a}{p^n}$, $w' = \frac{a}{p^{n'}}$, &c., then the expression

$$\frac{\log.(a+a'+a''+a'''+\&c.) - \log.(w+w'+w''+w'''+\&c.)}{\log. p}$$

gives the time required.

38. A person owes 40000*l.* As he is not able to pay it off at once, he agrees with his Creditors to pay, beginning from this day, at the end of each year, 2500*l.*, and 5 per cent interest on his debt. When he calculated the time which it would require to discharge the whole of the debt, he found, that at the end of the last year, he had not

2500*l.* to pay, but something less: how much less than this sum had he to pay, reckoning compound interest?

Ans. Less than this sum by 31.88*l.*

39. Required to find four numbers, such, that their sum $= a$, the sum of their squares $= b$, the sum of the 12 products, which arise from multiplying each number with the square of another, $= c$, and the sum of the 6 products, which arise from multiplying each number by the square of another, $= d$?

Ans. The sum of the products of the four numbers sought, taken two and two, $= \frac{1}{2}(a^2 - b)$, the sum of their products, taken three and three, $= \frac{1}{6}(a^3 - ab - 2c)$, and the product of all four $= \frac{1}{24}(a^4 + 2a^2b - 3b^2 - 8ac + 12d)$. The four numbers are consequently expressed by the following equation: $x^4 - ax^3 + \frac{1}{2}(a^2 - b)x^2 - \frac{1}{6}(a^3 - ab - 2c)x + \frac{1}{24}(a^4 + 2a^2b - 3b^2 - 8ac + 12d) = 0$.

40. Four numbers are given by the following conditions: their sum $= a$, the sum of their squares $= b$, the sum of their cubes $= c$, and the sum of their 4th powers $= d$. How are they found?

Ans. The sum of the products of the four numbers sought, taken two and two, $= \frac{1}{2}(a^2 - b)$, the sum of their products, taken three and three, $= \frac{1}{6}(a^3 - 3ab + 2c)$, and the product of all four $= \frac{1}{24}(a^4 - 6a^2b + 3b^2 + 8ac - 6d)$. The four numbers are consequently given by the following equation: $x^4 - ax^3 + \frac{1}{2}(a^2 - b)x^2 - \frac{1}{6}(a^3 - 3ab + 2c)x + \frac{1}{24}(a^4 - 6a^2b + 3b^2 + 8ac - 6d) = 0$.

FINIS.

T. C. Hansard, Pater-noster-row Press.

Gen^{l} quad^{c} $ax^2 + bx = c$

$4a^2x^2 + 4abx + b^2 = b^2 + 4ac$

$2ax + b = \pm\sqrt{b^2+4ac}$

$x = \dfrac{-b \pm \sqrt{b^2+4ac}}{2a}$

$$\frac{8940}{9017} = \frac{2.3.7.71}{71.127} = \frac{120}{127}$$

$$\frac{5970}{4177} = \frac{2.3.5.13.23}{3.7.19.23} =$$